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INVESTIGATION OF PROCEDURES FOR AUTOMATIC RESONANCE EXTRACTION FROM NOISY TRANSIENT ELECTROMAGNETICS DATA

Final Report for Contract # N00014-80-C-0299

lume I - Investigation of Resonance Extraction Procedures

to

Office of Naval Research
800 North Quincy Street
Arlington, VA 22217

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17 August 1981

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This volume presents major results and conclusions of research in resonance extraction techniques from transient electromagnetics data. Section 2 addresses the question of what the best methods of estimating the resonances for this application are. Among the estimation procedures studied are the conventional technique of using an inhomogeneous set of equations, eignvalue decomposition techniques, Jain's method, and iterative techniques. Section 3 studies procedures for selecting the proper order and eliminating extraneous →

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resonances. The maximum likelihood procedure for order selection is presented in Section 3. This procedure is capable of automatically determining the proper order without any prior knowledge of the noise level. Also, in Section 3, Henderson's procedure for eliminating extraneous resonances is illustrated by numerical example. The procedures presented in Section 3 allow an efficient, automatic procedure for order selection and resonance estimation to be constructed by using eigenvalue decomposition of the data. Section 4 discusses various practical problems that must be addressed before resonance extraction procedures for a practical radar target recognition system can be designed. This section serves to relate the concepts in Sections 2 and 3 to the application under consideration in the current contract. Section 4 addresses the basic problem of what is the best sampling rate and period of observation for the purposes of resonance extraction. Possible procedures and systems for both target recognition and target intelligence using natural resonances are presented. Also in Section 4, two key problems with Prony's method are identified. Methods for effectively treating these problems are presented which allow Prony's method to be effective at high noise levels.

INVESTIGATION OF PROCEDURES FOR AUTOMATIC RESONANCE EXTRACTION FROM NOISY TRANSIENT ELECTROMAGNETICS DATA

Final Report for Contract # N0004-80-C-0299

Volume I - Investigation of Resonance Extraction Procedures

by

Jon R. Auton
Michael L. Van Blaricum

17 August 1981

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1.0 INTRODUCTION

1.1 BACKGROUND

In analyzing and characterizing experimental or numerical electromagnetic response data, one generally desires to extract parameters that can be related to physical characteristics of the system being studied. One set of physically related parameters is the complex natural resonances of the system and their related coefficients. Indeed these resonances have been recognized as important for some time. Recently the concept of electromagnetic resonances has been applied to problems of particular interest to the Navy.

In 1971 Baum [1] developed the formalism known as the Singularity Expansion Method (SEM) which enables one to write any electromagnetic response of a system as an expansion of the complex resonances or poles and residues of that system. Baum developed this formalism for the electromagnetic pulse (EMP) community so that the external current distribution on aircraft and ground support systems could be characterized concisely. It was not until the singularity expansion method came into being that it was possible to determine the modal resonances and the excitation coefficients of each mode for a structure with an arbitrary incident excitation.

In 1974 Mains and Moffatt [2] introduced the concept of using the complex natural resonances of a body as a basis for target recognition. They made use of the fact that a few natural resonances of a body are adequate to distinguish the body within a finite collection of bodies. They also made use of the knowledge that the natural resonances of a body, as manifested in a scattered waveform, are aspect independent. The identification procedure which was used was to first obtain a set of multi-frequency radar scattering data from an unknown target. A prediction-correlator type of processing was then applied to select the real target from a catalog of candidate targets and their resonances.

In both Baum's and Mains and Moffatt's early work the resonances were obtained from a set of equations which characterized the electromagnetic response of the body, much as a circuit theorist finds his resonances by solving a differential equation. Many shapes have been numerically analyzed to date to obtain their characteristic resonances. However for complex shapes and configurations, it is of great interest to be able to obtain the resonances from experimental data. In particular obtaining the resonance parameters from transient response data from EMP simulators and transient radar ranges is of keen interest to the military community.

Spectral electromagnetic response data usually lends itself to the visual identification of these natural frequencies. The damping constants cannot be obtained as easily, however, and often are calculated from estimates of the quality factor (Q). Similarly, temporal response data generally allows one to visually determine the dominant natural frequency in a response. If enough data are present, the damping constant of this dominant frequency can also be determined. However, temporal data are usually Fourier transformed to the frequency domain so that the higher order modes can be identified visually. Visual identification of the natural resonances of a system is not ideal by any means, particularly if the system has many low Q modes. Hence a numerical resonance extraction procedure is very much in demand.

About seven years ago Prony's algorithm [3] was applied for the first time to the problem of numerically extracting the natural resonances from transient electromagnetics response data. The first application of Prony's method was to the numerically generated transient current on a thin dipole. The results, which were reported at the 1974 USNC/URSI meeting [4] by Mittra and Van Blaricum gave a set of resonances (poles) which compared very closely with the first ten even modes previously calculated by Tesche [5]. As a result of this initial demenstration, several researchers began studying Prony's method to determine its utility for analyzing several kinds of transient data and to look for solutions to some of the

problems inherent in the process. In addition, Brittingham, Miller and Willows [6] demonstrated that a procedure parallel to the time domain Prony's method could be applied to frequency domain data.

Some of the initial questions which were asked about Prony's method, and which are in part still begin studied, were:

1. Will Prony's method work if multiple poles are present?
2. How does one determine a priori the order of the system? That is, how many poles are contained in the response data?
3. What effects do noisy data have on the Prony algorithm?
4. How do we insure or know the accuracy of the poles returned?

These questions were all addressed to some extent in Van Blaricum's dissertation [7]. It was found that Prony's method would work for the case of multiple poles without any change in the pole searching algorithm. Two methods were discovered by which the order of the system could be determined. These methods are the Householder orthogonalization procedure and the Eigenvalue method. Examples of these methods can be found in Van Blaricum's thesis [7] and appear in the special EMP issue of AP-S [8]. A preliminary study of the effects of noise on the pole extraction algorithm and the order determination algorithms was presented in [7] and [8] also. However the problem of noise and Prony's method is a very complex one which has not been, even at this time, completely answered. Several alternatives to Prony's method have been suggested and investigated with the hope of finding a cure for all noise problems. Among these alternatives are variations on Prony's method [9-10]; the pencil-of-functions method [11] which is presented in Appendix D; iterative generalized least squares presented in Appendix A; column Prony's method, Appendix F; the adaptive method, Appendix C; and Evan's and Fischl's method, Appendix G.

To understand the different procedures for pole extraction and the difficulties involved it is necessary to discuss the main elements of the

existing procedures. The process of extracting the natural resonances and their associated residues from a transient signal has four main steps as shown in Figure 1.1.

The first step is the determination of the order of the system. At this step one decides how many poles the system response function has so that the proper model order can be obtained. It has been found, through a combination of parameter studies and trial and error, that if the order of the system is underestimated then the extracted poles will deviate substantially from the true poles. Similarly if the order of the system is overdetermined, the algorithm produces extraneous poles. The presence of the extraneous poles causes the residues of the true poles to be inaccurate and also results in unnecessary computation time. The presence of noise in data makes the determination of the system order a very complex problem. Up to this time many methods have been used to determine the order but they either break down when noise is present or they are dependent on trial and error or the intervention of the user. For analysis of massive amounts of data, as in the case of EMP data, or for radar target identification a totally automated method is a must. In Section 3 (Volume I) of this report, a procedure capable of automatically determining the proper order without any knowledge of the noise level is presented.

Once the order of the system has been determined the coefficients of the linear predictor equation or Prony's difference equation must be solved. The degree of difficulty of this step depends on the noise level in the data and on the proper determination of the order of the system. In Section 4, certain factors which greatly effect the accuracy of the coefficients are discussed.

Once the difference equation coefficients are obtained, the roots of an N^{th} order polynomial, N being the system order, must be found. Many root finding routines exist but Muller's method [13] appears to be

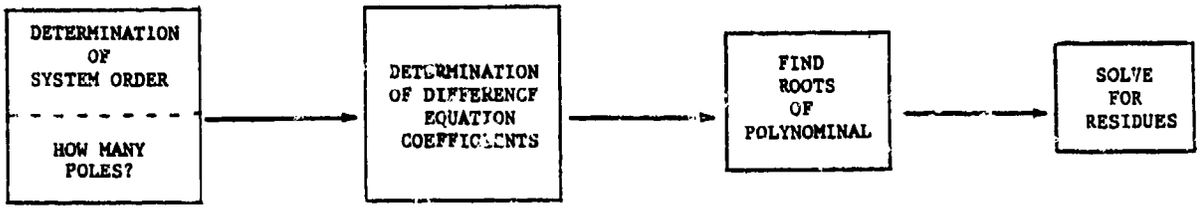

FIGURE 1.1. The Elements of the Extraction of Natural Resonances and Associated Residues from a Transient Signal

the optimal method. While this is a key step in the procedure it is totally dependent on the accuracy of the coefficients which were obtained in the previous step.

The final step is the solution of the residues which are associated with the system poles or singularities. These residues are obtained by solving a simple linear matrix equation. In many problems, such as target identification, the residues are not required and hence this is certainly not a critical step.

1.2 RECENT WORK IN RESONANCE ESTIMATION

Kulp [14] has recently studied the effect of sampling rate on the accuracy of Prony's method. The sampling rate, we have found, is one of two factors that determine the performance of Prony's method with noisy data. The other factor is statistical bias in the estimates. If both of these problems are treated, Prony's method performs admirably well with noisy data.

Cures for the bias problem and the problem of extraneous resonances have been proposed by Henderson[15]. The technique involves the use of eigenvalue decomposition to construct the coefficients of Prony's difference equation. Although we cannot prove, as yet, that the use of eigenvalue decomposition allows bias-free estimation of the coefficients, we have seen indications that this is the case. In Section 2, Volume I of this report, indications that lead us to this bias-free remark are described.

It is these two recent works plus order selection techniques developed by ETI that set the stage for testing of the automatic resonance extraction procedures in Phase II of the current contract.

1.3 OBJECTIVES AND CONTRIBUTIONS OF THE PRESENT WORK

Three major objectives were set forth at the beginning of the current contract:

1. To review and summarize the state-of-the-art of procedures for extracting resonances from transient data which include Prony's method, adaptations of Prony's method and other techniques of system identification and parameter estimation.
2. To investigate methods for automatically determining the proper order of a system represented by a noisy set of data and the limitations of the procedures for doing so.
3. To investigate methods for automatically extracting the proper system resonances from a set of noisy data once the proper order has been established.

At the present time these objectives have been fully satisfied. Our present understanding of the resonance extraction problem will enable us to construct and test practical automatic procedures for resonance extraction in the second phase of the current contract. In looking for automated methods, several other important issues such as biased estimates, iterative versus one shot processes, and pattern recognition, to name a few, were investigated. Many possible techniques were examined in a search for automated methods. These techniques are summarized in appendix form in Volume II of this report.

1.4 SOME REMOTE SENSING APPLICATIONS OF RESONANCE EXTRACTION

The ideal end product of this type of research is an automatic real time technique which will take noisy transient signals and estimate the complex resonances of the structure and the errors in these estimates. In studying existing techniques with this one end product in mind it became clear to us that the method developed is dependent on the specific uses intended.

The three major remote sensing applications of the complex resonances are target recognition, target intelligence, and target camouflage and decoying. Target recognition, as we define it, assumes that previous knowledge of the targets in question is known. Target recognition is also assumed to be a real-time or a near real-time process. The poles extracted from the measured target's signature can be compared to a target pole catalog for identification. While this requires work in building the initial library it makes the actual pole extraction procedures less complicated (fortunately since they have to be real time) because the true poles will not have to be separated from the noise poles.

Target intelligence assumes that we are trying to discover the shape and type of vehicle we are seeing having never seen it before. Hence this target will not be in a library or catalog. This requires that the true system poles be separable from the noise poles. In addition the relation between pole patterns or locations and the target's physical characteristics must be known. The process, however, does not have to be real-time and can usually have human input.

Target camouflage and decoying are potential techniques based on knowing the relationships between pole patterns and physical characteristics. Knowing these relationships a target can be given apparent (to the radar) new characteristics by modifying its measured pole pattern. Hence one could make a cruise missile either apparently disappear by overdamping its resonant return or make it look like another vehicle by appropriately moving the resonances. Target camouflage is much like target intellegence in its requirements. The main requirement is developing the knowledge of the relationship among the poles and physical configuration.

Before much more work is done in this general theoretical area of resonance extraction from transient signals it will be necessary to very accurately define the desired type of system application to which the methods will be applied. In Section 4, Volume I of this report, we define tentative and very rough forms of the procedures to be applied to

the target recognition and target intelligence problems. These rough forms will gain further definition in the second phase of the current contract.

1.5 OUTLINE OF THIS REPORT

Volume I presents major results and conclusions of the research. Section 2 addresses the question of what the best methods of estimating the resonances for this application are. Among the estimation procedures studied are the conventional technique of using an inhomogeneous set of equations, eigenvalue decomposition techniques, Jain's method, and iterative techniques. Section 3 studies procedures for selecting the proper order and eliminating extraneous resonances. One major and original contribution by ETI is presented in Section 3, namely, the maximum likelihood procedure for order selection. This procedure is capable of automatically determining the proper order without any prior knowledge of the noise level. Also, in Section 3, Henderson's procedure for eliminating extraneous resonances is tested by numerical example. The procedures presented in Section 3 allow an efficient, automatic procedure for order selection and resonance estimation to be constructed by using eigenvalue decomposition of the data. Section 4 discusses various practical problems that must be addressed before resonance extraction procedures for a practical radar target recognition system can be designed. This section serves to relate the concepts in Sections 2 and 3 to the application under consideration in the current contract. Section 4 addresses the basic problem of what is the best sampling rate and period of observation for the purposes of resonance extraction. Possible procedures and systems for both target recognition and target intelligence using natural resonances are presented. Also in Section 4, two key problems with Prony's method are identified. Methods for effectively treating these problems are presented which allow Prony's method to be effective at high noise levels.

Volume II consists of appendices which present the details of several techniques that have been investigated during this contract. Volume I makes references to Volume II for the details of specific procedures.

Volume III contains a large bibliography of Prony's method. Also in Volume III is a translation of Prony's original paper.

2.0 COMPARISON OF PROCEDURES FOR RESONANCE ESTIMATION

2.1 DEFINITION OF THE RESONANCE EXTRACTION PROBLEM

In this report we consider the problem of estimating the resonances of a linear, time-invariant, physical system, such as an electromagnetic scatterer, from the measured time-domain system response to the measured time-domain excitation. We assume that a distributed physical system such as a scatterer can be adequately modeled by a single-input, single-output linear system where the input may be the incident electric field time history and the output is the reflected electric far-field time history.

In general, both the excitation and response are required to estimate the resonances of the system. However, for the first part of this section we consider the case where the excitation is a delta function so that the impluse response can be observed. In practical situations the impulse response can usually be obtained by deconvolution. (Appendix H presents one deconvolution technique.) The measured impulse response can be expressed as

$$q_i = w_i + e_i = \sum_{j=1}^{k} A_j \exp(s_j\, i\Delta t) + e_i = \sum_{j=1}^{k} A_j\, z_j^{\,i} + e_i \qquad (1)$$

for $i=0,1\ldots,N_s$ where $z_j = \exp(s_j\Delta t)$ are the resonances or poles in terms of z-transform variable, the s_j are the poles in terms of the Laplace-transform variable, the A_j are termed "residues", q_i denotes the i^{th} sample, N_s is the number of samples, and Δt is the time increment between sucessive samples. Equal increments are assumed.

Distributed systems normally have an infinite number of modes, i.e., k=∞. In practice, the number of poles that are modeled, n, need not be infinite but only sufficiently large to yield an adequate approximation to the measured response. The "best" value of n is strongly dependent on the number of dominant resonances and the noise level.

Many procedures are available for estimating the resonances. But, for the first part of this section, we consider only the class of procedures based on Prony's difference equation which has the form

$$\sum_{j=0}^{n} \alpha_j w_{j+i} = 0, \quad i=0,1,\ldots,m. \tag{2}$$

where w_i denotes the unocrrupted value of q_i. If w_i is replaced by q_i the equation becomes

$$\sum_{j=0}^{n} \alpha_j q_{j+i} = d_i, \quad i=0,1,\ldots,m. \tag{3}$$

where d_i is, in general, nonzero, and is termed the "equation error". In matrix form (2) becomes Wx=0 and (3) becomes Qx=d

where $Q = \begin{bmatrix} q_0 & q_1 & \cdots & q_n \\ q_1 & q_2 & \cdots & q_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ q_m & q_{m+1} & \cdots & q_{m+n} \end{bmatrix}$ and $W = \begin{bmatrix} w_0 & w_1 & \cdots & w_n \\ w_1 & w_2 & \cdots & w_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_m & w_{m+1} & \cdots & w_{m+n} \end{bmatrix}$

are the (M x N)-dimensional data matrices, $x = [\alpha_0,\alpha_1,\ldots,\alpha_n]^T$ is the N-dimensional parameter vector, and $d = [d_0,d_1,\ldots,d_m]^T$ is the M-dimensional equation error vector. For convenience we use M=m+1 and N=n+1.

The central problem of all noniterative resonance extraction procedures is choosing the parameter vector to minimize the magnitude of the equation error. Various procedures for minimizing d are available. In this section, we examine some of these procedures and isolate one procedure that is the most appropriate for resonance extraction.

Once x has been adjusted to minimize d, the roots z of the equation

$$\sum_{j=0}^{n} \alpha_j z^j = 0 \tag{4}$$

are estimates of the resonances in the z-plane and define estimates of the s-plane resonances through $s_j' = \ell n z_j'/\Delta t$, for j=1,...,n where, for the moment, n=k is assumed. Estimates of the residues can be taken as the coefficients A_j' that minimize the "true error" defined as

$$\sum_{i=0}^{m+n} [q_i - \sum_{j=0}^{n} A_j' (z_j')^i]^2 . \tag{5}$$

The minimizing A_j' can be chosen by using a standard least-squares technique.

An important observation can be made from (4): If the α_j are all multiplied by the same constant, the roots, z_j', are unchanged. From this observation it is concluded that the magnitude of the x vector is irrelevant in estimating the resonances. Only the direction of x is important.

In this section we assume that the true number of poles,k, is finite and known and that n=k. The section on order selection procedures discusses what courses of action are available when k is not known and possibly infinite. The presence of measurement error produces a data matrix of rank N while the true rank without measurement error is k. Measurement error, then, greatly complicates estimation of the true order.

2.2 QUADRATIC ERROR MINIMIZATION

One procedure for minimizing the equation error is straight minimization of its norm, $||d||$, or, equivalently, the minimization of the square of the norm, $d^T d$. We refer to $d^T d$ as quadratic error (QE). Since $d=Qx$, the trivial solution $x=0$ will always provide the minimum QE. Other values of x can provide this same minimum value only if the rank of Q is less than N which occurs only in the absence of measurement error. Nevertheless, we may proceed in the usual fashion to derive a set of normal equations by equating the derivative of $d^T d$ with respect to x to zero:

$$\nabla_x (d^T d) = 2Q^T Qx = 0$$

which implies that the value of x that satisfies $Q^T Q\ x=0$ can furnish a minimum. Unfortunately, when measurement error is present $Q^T Q$ is nonsingular, and hence, only the trivial solution, $x=0$, exists as we noted above. The trivial solution is not desirable since the zero vector contains none of the directional information that is necessary to estimate the resonances.

One way out of this dilemma is to fix one element of the x vector at a nonzero value and adjust the remaining elements to minimize $d^T d$. Suppose we let $\alpha_n=1$. Then $Qx=d$ can be written $\bar{Q}\bar{x}+\bar{q}=d$ where $\bar{x} \equiv [\alpha_0, \alpha_1, \ldots \alpha_{n-1}]^T$,

$$\bar{Q} = \begin{bmatrix} q_0 & q_1 & \cdots & q_{n-1} \\ q_1 & q_2 & \cdots & q_n \\ \vdots & \vdots & & \vdots \\ q_m & q_{m+1} & \cdots & q_{m+n-1} \end{bmatrix} ,$$

and $\bar{q}=[q_n, q_{n+1}, \ldots q_{m+n}]^T$. The deriviative of $d^T d$ with respect to $\bar{x}$ is

$$\nabla_{\bar{x}} (d^T d) = \nabla_{\bar{x}} [(\bar{\bar{Q}}\bar{x} + \bar{q})^T (\bar{\bar{Q}}\bar{x} + \bar{q})]$$
$$= 2\bar{\bar{Q}}^T\bar{\bar{Q}}\bar{x} + 2\bar{\bar{Q}}^T\bar{q} \ .$$

Equating the deriviative to zero implies that

$$\bar{x}_I = - [\bar{\bar{Q}}^T\bar{\bar{Q}}]^{-1}\bar{\bar{Q}}^T\bar{q} \ .$$

With or without measurement error Q^TQ is always nonsingular, and hence, its inverse exists. It follows that $\bar{x}_I$ can always be used to find a nontrivial parameter vector. $\bar{x}_I$ is referred to as the reduced or inhomogeneous solution because it satisfies the inhomogeneous equation: $\bar{\bar{Q}}^T\bar{\bar{Q}}\bar{x}_I = \bar{\bar{Q}}^T\bar{q}$.

The use of the inhomogeneous solution has traditionally been the procedure for obtaining a parameter vector. We will examine this solution procedure more closely after another procedure is introduced.

2.3 MINIMIZATION OF NORMALIZED QUADRATIC ERROR

We have seen that d^Td becomes zero when x^Tx is zero which leads to the problem of the trivial solution. An error criterion that does not necessarily tend to zero when the parameter vector tends to zero is the normalized quadratic error (NQE) defined as d^Td/x^Tx. The deriviative of d^Td/x^Tx with respect to x is

$$\nabla_x \frac{d^Td}{x^Tx} = \frac{2Q^TQx}{x^Tx} - 2x \frac{d^Td}{(x^Tx)^2} \ .$$

Equating the deriviative to zero implies that

$$Q^TQx = \frac{d^Td}{x^Tx} x \ .$$

This is the well-known form of an eigenvalue problem. The only values of x capable of furnishing a minimum of NQE are the eigenvectors of Q^TQ. NQE assumes the minimum as the lowest eigenvalue when x assumes the value of the weakest eigenvector. We conclude that minimizing NQE effectively avoids the trivial solution and is equivalent to using an eigenvalue technique to find a minimizing parameter vector. The weakest eigenvector may be used to estimate the resonances. Price [9] arrived at this result using different but equivalent arguments.

2.4 SINGULAR VALUE/EIGENVALUE DECOMPOSITION

Another way of representing the eigenvalue analysis of Q^TQ is the eigenvalue decomposition (EVD) of Q^TQ which can be expressed as

$$Q^TQ = V\Lambda_N V^T$$

where V is an (N x N)-dimensional orthogonal matrix whose columns are the eigenvectors of Q^TQ and Λ_N is an (N x N)-dimensional diagonal matrix whose diagonal elements are the eigenvalues of Q^TQ.

In the same way, the eigenvalue analysis of QQ^T can be expressed in the EVD form:

$$QQ^T = U\Lambda_M U^T$$

where U is an (M x M)-dimensional orthogonal matrix whose columns are the eigenvectors of QQ^T and Λ_M is an (M x M)-dimensional diagonal matrix whose elements are the eigenvalues of QQ^T.

A more general form of EVD for nonsquare matrices, known as singular value decomposition [16] (SVD), effectively performs both EVD's mentioned above. The SVD of Q is expressed as

$$Q = USV^T$$

where U and V are the matrixes that were previously defined and S is an (M x N)-dimensional matrix whose diagonal elements are non-negative and are called the singular values of Q. The singular values are related to the eigenvalues by $\Lambda_N = S^TS$ and $\Lambda_M = SS^T$.

2.5 USE OF EVD/SVD TO APPROXIMATE THE DATA MATRIX

We have seen that straight minimization of d^Td leads to $Q^TQ=0$ which for the noisy case has only the trivial solution since Q^TQ is nonsingular. Two methods to avoid the trivial solution, fixing an element of the parameter vector and EVD/SVD analysis, have been mentioned. Another interpretation of EVD/SVD analysis promotes understanding of what these methods accomplish. EVD/SVD analysis can be interpreted as a method to approximate Q^TQ or Q with a matrix of lower rank or, in other words, a singular matrix. When Q^TQ is replaced by a singular approximant matrix, the homogeneous equation mentioned above has a nontrivial solution. It is shown that this nontrivial solution is equivalent to the weakest eigenvector of Q^TQ.

A singular approximant of $P=Q^TQ$ can be constructed by using the EVD of Q^TQ as

$$\hat{P}_s = V\Lambda_{NS}V^T$$

where Λ_{NS} is the diagonal matrix constructed from Λ_N by forcing the smallest diagonal element or eigenvalue to zero. The eigenvector corresponding to the smallest nonzero eigenvalue must satisfy $\hat{P}_s x=0$ and hence, it is the nontrivial solution to the best approximant of the homogeneous equation $Q^TQx=0$. $\hat{P}_s$ is the optimal approximant to P in the sense that is the singular matrix, P_s, that minimizes the euclidean matrix norm of P_s-P.

An identical procedure is the use of the SVD of Q to construct an approximant of Q:

$$\hat{Q}_s = US_sV^T$$

where S_s is constructed from S by forcing the smallest singular value to zero. Since $\hat{P}_s = \hat{Q}_s^T\hat{Q}_s$ the resulting solution is identical to the solution obtained with EVD of Q^TQ.

One important observation should be made about the approximation $\hat{Q}_s$: This matrix is not a Hankel matrix in general. Therefore, we cannot use this matrix to form an approximation to the original waveform.

2.6 SUPERIORITY OF EVD/SVD TECHNIQUES FOR RESONANCE ESTIMATION

So far two methods to find a minimizing parameter vector have been found:

1. The inhomogeneous solution
2. EVD/SVD analysis

In the following we show that the directions (not just the magnitudes) of the parameter vectors obtained using these two procedures are different, and hence, the estimates of the resonances differ. We also show that in one sense EVD/SVD analysis is superior to the inhomogeneous solution.

Let x_{NQE} be the weakest eigenvector of Q^TQ which has been scaled so that $\alpha_n=1$. (Note that scaling the eigenvector does not change its direction, and hence, the estimates of the resonances are unchanged.) Let $\bar{x}_{NQE}$ denote the reduced parameter vector which is formed from x_{NQE}. Then $Qx_{NQE} = \bar{Q}\bar{x}_{NQE} + \bar{q} = d_{NQE}$. By taking the pseudoinverse of $\bar{Q}$ we solve for $\bar{x}_{NQE}$:

$$\bar{x}_{NQE} = -[\bar{Q}^T\bar{Q}]^{-1}\bar{Q}^T\bar{q} + [\bar{Q}^T\bar{Q}]^{-1}\bar{Q}^T d_{NQE} \quad .$$

Since the inhomogeneous solution is defined as $\bar{x}_I = -[\bar{Q}^T\bar{Q}]^{-1}\bar{Q}^T\bar{q}$, the difference between the two estimates is

$$\bar{x}_{NQE} - \bar{x}_I = [\bar{Q}^T\bar{Q}]^{-1}\,\bar{Q}^T d_{NQE} \quad .$$

When noise is present, this difference can be shown to be, in general, nonzero. Since $\alpha_n = 1$ for both procedures, the parameter vectors constructed from these reduced vectors must differ in direction in the general case.

Now that the two procedures have been shown to produce different resonance estimates, we provide an argument to show that x_{NQE} is the better of the two estimates because it is perturbed less than the inhomogeneous solution by the presence of noise in the data.

If we assume that the noise corrputing the waveform is zero-mean, Gaussian-distributed, and uncorrelated with variance σ^2 then we can show that [7]

$$\xi(Q^TQ) = W^TW + IM\sigma^2 \ ,$$
$$\xi(\bar{Q}^T\bar{Q}) = \bar{W}^T\bar{W} + IM\sigma^2 \ ,$$

and $\xi(\bar{Q}^T\bar{q}) = \bar{W}^T\bar{w}$ where I denote the identity matrix of appropriate dimension in each case, $\bar{W}$ and $\bar{w}$ are the uncorrupted matrices corresponding to $\bar{Q}$ and $\bar{q}$, respectively, ξ denotes the expectation operator, and M is the number of rows of $\bar{Q}$ and Q.

In order to perform a completely deterministic analysis of the effects of noise on each procedure, we replace Q^TQ, $\bar{Q}^T\bar{Q}$ and $\bar{Q}^T\bar{q}$ with their expectations we derived above.

The EVD of W^TW is denoted by $W^TW = V^W\Lambda_N^WV^{WT}$. Then the EVD of $\xi[Q^TQ]$ is

$$\xi[Q^TQ] = V^W[\Lambda_N^W + IM\sigma^2]V^{WT}$$

Since W^TW and $\xi(Q^TQ)$ possess the same eigenvectors, we conclude that noise has no net effect, on the average, on the eigenvectors of Q^TQ including x_{NQE}. Proof is not yet available to support the claim that the expectation of the eigenvectors is uneffected by noise. However, we provide the following geometrical argument: The eigenvectors represent the extremes of the directions present in the data, that is, either the strongest or weakest directions are represented by eigenvectors. A purely random noise component has no dominant directions and tends to perturb the eigenvectors equally in all directions neighboring their unperturbed or noise-free directions. Because of the symmetry in the perturbations, the average or expectation of the perturbed eigenvectors should be equal to the unperturbed eigenvectors themselves.

Let $\bar{x}_{IE}$, which is referred to as the expected inhomogeneous solution, satisfy

$$\xi(\bar{Q}^T\bar{Q})\ \bar{x}_{IE} = \xi(\bar{Q}^T\bar{q})$$

$$\text{or} \quad (\bar{W}^T\bar{W} + I_nM\sigma^2)\ \bar{x}_{IE} = \bar{W}^T\bar{w}\ . \tag{6}$$

We assume that the expected solution is representative of actual solutions, $\bar{x}_I$.

The following observations can then be made:

1. When $\sigma=0$, i.e., no noise, the expected solution reduces to the noise-free or exact solution.

2. When the noise greatly dominates the signal we have $\bar{x}_{IE} \approx 0$.

3. From 1 and 2, the expected solution is highly dependent on the noise level.

The third observation contrasts the effects of noise on the inhomogeneous solution and on x_{NQE}.

From the foregoing analysis we conclude the following:

1. x_{NQE} and $\bar{x}_I$ lead to different resonance estimates.

2. Resonance estimates from $\bar{x}_I$ are considered to be inferior to those from x_{NQE} because $\bar{x}_I$ is expected to display a strong dependence on the noise level while x_{NQE} is expected to remain invariant with the noise level. In the extreme case when the noise greatly dominates the signal $\bar{x}_I$ tends to the trivial solution: $\bar{x}_I \to 0$.

Therefore this analysis supports the claim that the eigenvectors are perturbed less by noise than the inhomogeneous solution. We point out that the validity of this analysis critically depends on the validity of the assumption that the results for expected or average cases are representative of results for actual cases. However, in the following numerical example, actual cases are presented that support the conclusion that the eigenvectors are perturbed less by noise.

2.7 NUMERICAL EXAMPLE

Figure 2.1 depicts the waveform that is used in this numerical example. The poles, s_j, and residues, A_j, which define the uncorrupted waveform are given in Table 2.1. The true number of resonances is k = 12. The time step is $\Delta t = 0.2$. The waveform consists of 400 samples.

Table 2.2 displays the parameter vector obtained with the inhomogeneous solution for the case when n=k and M=387 and for various noise levels. When the noise becomes large all parameters except α_n tend toward zero. Table 2.3 displays the weakest eigenvector, x_{NQE}, for N=13 and M=387 and for the same noise levels. The parameters in this case show no tendency toward zero. This example, then, supports the conclusion that the weakest eigenvector is effected less by noise than the parameter vector constructed from the inhomogeneous solution.

Appendix I presents a technique to discriminate between the true poles and the extraneous poles for the case when n>k. This technique makes use of the tendancy of extraneous poles to remain in the left half of the s-plane when the waveform is reversed while the true poles are negated or "mirrored" through the imaginary axis of the s-plane. This tendency of the extraneous poles is closely related to the coefficients tending to zero in this example. It follows that this behavior is not observed if the weakest eigenvector is used to estimate the resonances. Appendix I describes in detail the reasons behind this behavior.

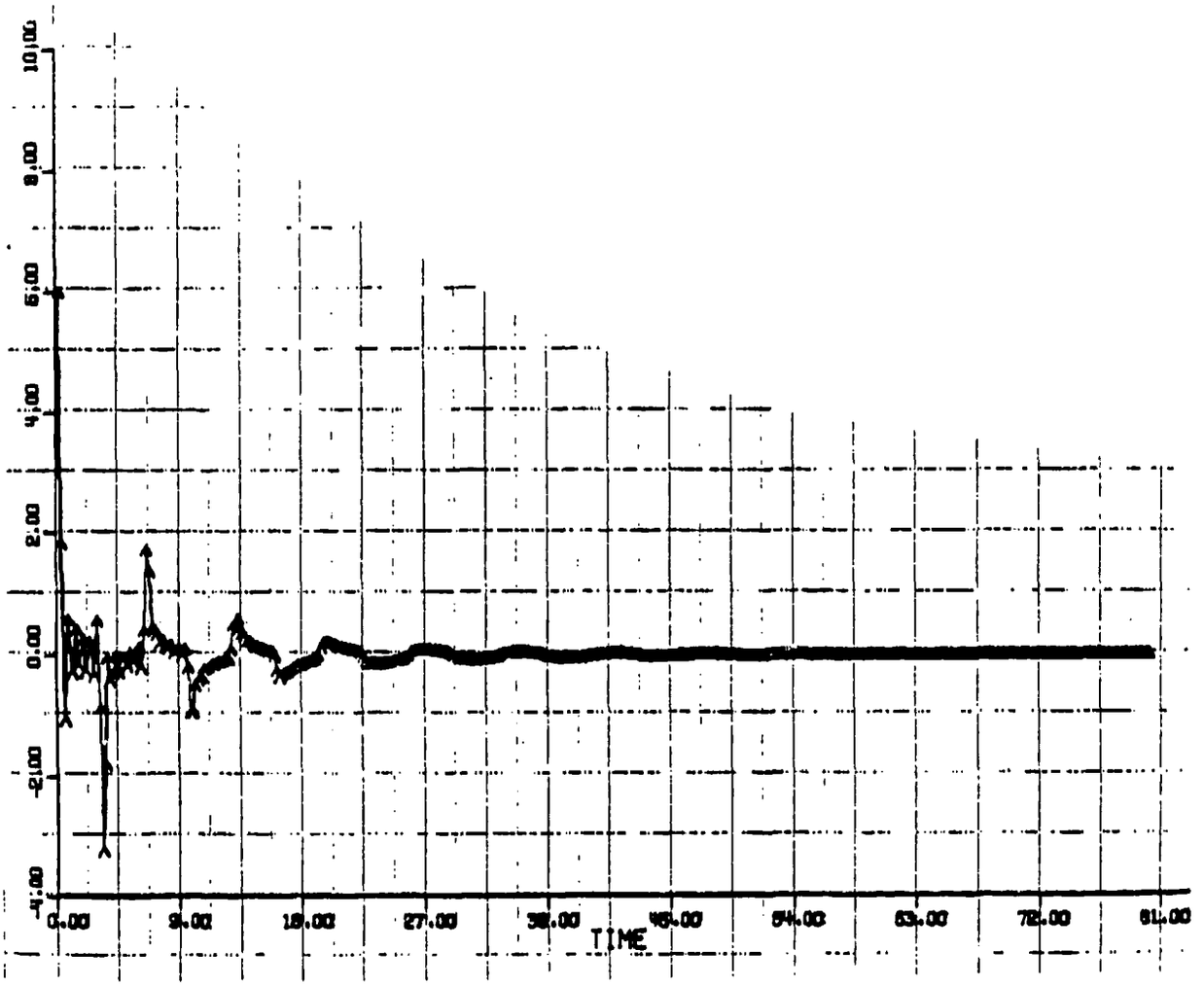

Figure 2.1 Waveform used in numerical example.

Table 2.1. Resonances for uncorrupted waveform with associated residues. Resonances are expressed in terms of s-plane poles.

j	s_j		A_j	
	REAL	IMAG.	REAL	IMAG.
1	-0.082	0.926	0.5	0.0
2	-0.082	-0.926	0.5	0.0
3	-0.147	2.874	0.5	0.0
4	-0.147	-2.874	0.5	0.0
5	-0.188	4.835	0.5	0.0
6	-0.188	-4.835	0.5	0.0
7	-0.220	6.800	0.5	0.0
8	-0.220	-6.800	0.5	0.0
9	-0.247	8.767	0.5	0.0
10	-0.247	-8.767	0.5	0.0
11	-0.270	10.733	0.5	0.0
12	-0.270	-10.733	0.5	0.0

Table 2.2 Coefficients for Prony's difference equation that are constructed from the inhomogeneous solution.

	a_j			
j	$\sigma = 0.001$	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1.0$
0	0.3189	-0.3082	0.1452	0.0384
1	-1.371	0.8076	-0.0868	-0.0211
2	3.384	-1.216	0.0743	-0.0112
3	-6.193	1.442	0.0257	0.0007
4	9.357	1.251	-0.1543	-0.0393
5	-12.17	0.6396	0.2438	0.0611
6	13.91	0.3044	-0.2537	-0.0245
7	-14.08	-1.355	0.1276	-0.0607
8	12.56	2.196	0.0177	-0.0007
9	-9.724	-2.628	-0.3176	0.0192
10	6.294	2.450	0.4166	-0.0865
11	-3.185	-1.914	-0.7654	-0.0861
12	1.000	1.000	1.000	1.000

Table 2.3 Coefficients for Prony's difference equation that are constructed by scaling the weakest eigenvector of $Q^T Q$.

j	α_j			
	$\sigma = 0.001$	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1.0$
0	0.6327	0.6372	-0.6416	-0.9769
1	-2.403	-2.453	1.544	0.8649
2	5.513	5.689	-2.378	1.275
3	-9.668	-10.03	3.024	-1.792
4	14.13	14.68	-2.995	-0.4695
5	-17.87	-18.58	2.218	2.061
6	-19.90	20.69	-0.7908	-0.5889
7	-19.62	-20.37	-0.9067	-1.924
8	17.01	17.64	2.369	1.644
9	-12.73	-13.16	-3.192	0.9120
10	7.876	8.118	3.103	-1.868
11	-3.686	-3.766	-2.315	0.1753
12	1.000	1.000	1.000	1.000

2.8 INTERPRETATION OF NONITERATIVE ESTIMATION METHODS IN TERMS OF THE EVD/SVD ANALYSIS OF THE DATA MATRIX

Now that the sense in which the EVD/SVD analysis is superior in resonance extraction has been defined, we proceed to soften this claim of superiority by demonstrating that all methods examined thus far can be interpreted as methods for constructing the parameter vector as a weighted combination of the eigenvectors of Q^TQ. The particular mode of combination and the particular choice of weights uniquely determine each method. Hence all methods in the end can be interpreted as particular methods in which EVD/SVD analysis plays an integral part. The question of superiority reduces to the question of what is the best way to combine the eigenvectors to form a parameter vector. Although we may not come to a conclusive answer to this question in this report, we examine several first approaches for the best method of combination in the section on order selection procedures.

Here we will show that our previous claim of superiority of EVD/SVD analysis only holds in a rather restricted case and restricted sense, and that an unqualified claim of superiority in the general case is not wise.

The inhomogeneous solution can be expressed as a linear combination of the eigenvectors of Q^TQ. To show this, we begin by noting that by Cramer's rule it is possible to write

$$\bar{x}_I = \frac{1}{\Delta_{NN}} [\Delta_{1N}, \Delta_{2N}, \ldots, \Delta_{nN}]^T$$

where Δ_{ij} denotes the element of the i^{th} row and j^{th} column of adj Q^TQ. Also we have the identity

$$\begin{aligned} \text{adj } Q^TQ &= \det Q^TQ \; V\Lambda_N^{-1}V^T \\ &= \det Q^TQ \sum_{i=1}^{N} \frac{v_i v_i^T}{\lambda_i} \end{aligned} \tag{7}$$

where $v_i \equiv$ the i^{th} column of $V \equiv$ the i^{th} eigenvector of Q^TQ corresponding to λ_i. A scaled parameter vector constructed from $\bar{x}_I$ which yields identical pole estimates is

$$x_I = [\Delta_{1N}, \Delta_{2N}, \ldots, \Delta_{NN}]^T$$

i.e., the N^{th} column of adj Q^TQ. From (7) it follows that x_I is a linear combination of the eigenvectors:

$$x_I = \sum_{i=1}^{N} C_i \, v_i$$

where the weighting coefficients are $C_i = \det Q^TQ \; v_{iN}/\lambda_i$ and v_{iN} is the element of the i^{th} row and the N^{th} column of V. Therefore, the estimates obtained with the inhomogeneous solution are equivalent to the estimates obtained using a weighted combination of the eigenvectors. Note that the eigenvectors are weighted in inverse proportion to the corresponding eigenvalues.

In the so-called EVD/SVD method of obtaining the parameter vector the weighting coefficients are all zero except the coefficient corresponding to the weakest eigenvector of Q^TQ which is non-zero. This method chooses the weakest eigenvector as the "natural" approximation to the eigenvector corresponding to the zero eigenvalue for the noise-free case.

The sensitivity of x_I to noise level is a particular function of the way the eigenvectors are weighted to form x_I. For the noise-free case, all coefficients except the one for the weakest eigenvalue are zero and x_I is simply a scaled version of the weakest eigenvector of Q^TQ. However, for the noisy case x_I is not determined only from the weakest eigenvector but is a combination of all eigenvectors. Since the eigenvectors span the space of all possible parameter vectors, x_I then has components that may not be appropriate for resonance estimation. These inappropriate perturbation of x_I are maifested in the expected sensitivity of x_I to the noise level. We stress that the components that are appropriate for resonance extraction have no clear definition at this time. Any absolute claim of superiority of one method over another is, in our opinion, ill-advised.

So far we have only considered the case when n = k and W^TW has only one zero eigenvalue. In this case, it seems quite appropriate to choose the weakest eigenvector of Q^TQ as the "natural" approximation to the weakest eigenvector of W^TW which is the true parameter vector in the noise-free case. But what is the appropriate choice when k < n? In this case, W^TW has more than one zero eigenvalue so that it seems appropriate to combine, in some way, the weakest n - k eigenvectors of Q^TQ to form a parameter vector. But what is the best method of combination? Also, it is necessary to reduce the dimensionality of the parameter vector when k < n since there are only k resonances and an N-dimensional parameter vector yields n = N-1 resonance estimates. How can this be done? These questions will be explored further in the section on order selection procedures.

2.9 JAIN'S METHOD

Another interesting method to find a parameter vector was developed by Jain and Gupta [17]. Although Jain uses this method with the pencil-of-functions procedure [18], we examine this method in the context of Prony's difference equation. The pencil-of-functions method is described in Appendix D. Jain's method consists of choosing the elements of x as the square roots of the diagonal elements of the adjoint matrix of Q^TQ. The choice of signs for the square roots was not a problem in the original pencil-of-functions method since for a large class of problems the square roots were known a priori to be positive. However, we are not as fortunate with Prony's difference equation: there is no means to establish the signs of the square roots a priori. Nevertheless, it is instructive to overlook this difficulty and to examine how the method may be interpreted in terms of the EVD of Q^TQ. Jain's method can be interpreted as a means of constructing a parameter vector from the eigenvectors of Q^TQ just as the previous two methods were interpreted.

Jain's method chooses the parameter vector as

$$x_J = \left[\pm\sqrt{\Delta_{11}},\ \pm\sqrt{\Delta_{22}},\ \ldots,\ \pm\sqrt{\Delta_{NN}} \right]^T \quad .$$

where the choice of sign is somehow known a priori. From (7) the diagonal elements of adj Q^TQ can be expressed as

$$\Delta_{ii} = \sum_{j=1}^{N} \frac{\det Q^TQ}{\lambda_j} v_{ij}^2$$

where v_{ij} denotes the element of the i^{th} row and j^{th} column of V. It is clear then that x_J is a combination of all the eigenvectors even though it is not a linear combination. Again, each eigenvector is weighted in inverse proportion to the corresponding eigenvalue as was done in the combination for x_I. It follows that x_J then is influenced by all the eigenvectors which may not be appropriate for resonance extraction. What effect the mode of combination has on x_J is difficult to estimate.

Because of the weights used in the combination for x_J, we would expect x_J to yield comparable resonance estimates to those yielded by x_I although these estimates would be different.

2.10 ITERATIVE TECHNIQUES

One can use EVD/SVD of Q^TQ to construct parameter vectors and estimate the resonances. But to obtain a fit to the response, coefficients A_j' must be chosen to minimize (5). This two-step procedure is peculiar. It would seem that both the z_j' and A_j' should be adjusted simultaneously for the best fit to the waveform. Henderson [15] has shown that for the noisy case the resonance estimates obtained by choosing the parameter vector as the weakest eigenvector of Q^TQ do not provide the absolute minimum of (5) or, in other words, they do not provide an optimal fit to the data in sense of (5). The basis of this behavior can be traced to the fact that the weakest eigenvector minimizes NQE. Minimizing NQE is not equivalent to minimizing (5).

To the authors' knowledge, only iterative procedures, such as that described by Evans and Fischl [19], (also described in Appendix G) can be used to choose both A_j' and z_j', simultaneously to minimize (5). Hence, if we wish to "improve" the estimates beyond those obtained with the EVD/SVD of Q^TQ we must resort to an iterative technique. The question of how much the choice of error criterion alters the estimates can be approached with a relation between the two criteria obtained by Evans and Fischl [19]:

$$d = X^T r \tag{8}$$

where

$$X = \begin{bmatrix} \alpha_0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & 0 \\ \vdots & \vdots & \cdots & \alpha_0 \\ \alpha_n & \alpha_{n-1} & & \vdots \\ 0 & \alpha_n & & \vdots \\ \vdots & \vdots & & \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \cdots & \alpha_{n-1} \\ 0 & 0 & & \alpha_n \end{bmatrix}$$

is an $(M + N - 1 \times M)$-dimensional matrix and $r = [r_0, r_1, \ldots, r_{m+n}]^T$ is the residual vector obtained by adjusting only the A_j' to minimize (5) for a given set of z_j' corresponding to a given parameter vector:

$$r_i = q_i - \sum_{j=0}^{n} A_j' (z_j')^i, \quad i = 0,1,\ldots,m + n .$$

From (8) we observe that minimizing the norm of r provides a minimum upper bound on the norm of d through

$$||d|| \leq ||X^T|| \, ||r|| \quad .$$

where $||X^T||$ is defined as the largest singular value of X^T. Since the characteristics of the X matrix depend on the data, we expect that certain extreme cases can be found where the two minimizations yield radically different estimates. Our own experience with actual data indicates that the two procedures yield comparable results when care is taken to apply each technique in the proper fashion. It is clear, however, that minimizing one error loosely minimizes the other so that if we are not too critical we may claim that both errors are approximately minimized simultaneously. Unfortunately it is not possible to predict how loosely the two minimizations are coupled without knowledge of the particular data given for resonance extraction.

Most iterative techniques attempt to minimize (5). Another novel iterative method, known as the iterative generalized least-squares procedure, does not attempt to minimize (5) but attempts to remove the source of "asympotic bias" of the parameter vector by "whitening the residuals". On pages 214-219 of Eykhoff [20] the source of the bias is defined as correlated residuals. On pages 244-247 the generalized least-squares method is defined and an algorithm due to Clarke [21] is presented as a means of implementing the method. Clarke's algorithm is specialized and applied to "whiten the residuals" of the least-squares Prony technique in Appendix A of this report. The generalized least-squares procedure is fundamentally attempting to minimize the equation error and is not attempting to minimize (5). Furthermore, the procedure attempts to neutralize asymtotic bias which is only defined for an infinite data sequence in the strictest interpretation. For these reasons, the generalized least-squares procedure is fundamentally different from the other type of iterative methods we have discussed.

For the inhomogeneous solution, which Clarke's procedure begins with, the bias is due the presence of a second-order noise term, $\bar{E}^T\bar{E}$, in $\bar{Q}^T\bar{Q} = (\bar{W} + \bar{E})^T(\bar{W} + \bar{E}) = \bar{W}^T\bar{W} + \bar{W}^T\bar{E} + \bar{E}^T\bar{W} + \bar{E}^T\bar{E}$. This same term is the source of the $M\sigma^2 I$ term in (6) which was held to cause the noise level sensitivity of $\bar{x}_I$. Perhaps the most significant comparison that can be made is between the generalized least-squares procedure and choosing the weakest eigenvector as the parameter vector. Both of these estimation techqniques neutralize the ill effects caused by the second-order noise term mentioned above. Although they will, in general, yield different estimates, the estimates are expected to be comparable in accuracy.

One major drawback of the iterative procedures is that they often do not converge. Guaranteed convergence is a requirement for any automatic resonance extraction procedure. We, therefore, recommend that the iterative techniques be avoided in this application. The use of EVD/SVD analysis to form resonance estimates is the safest procedure and can achieve the degree of accuracy of the iterative techniques.

2.11 GENERALIZATIONS

Up to this point, we have considered only the case that satisfies the following restrictions:

1. Prony's difference equation is used to estimate the resonances.
2. The impulse response of the system can be measured.

Now the more general case where these restrictions have been relaxed is considered. The generalization is accomplished by replacing Prony's difference equation with the equation

$$\sum_{j=0}^{n} (\beta_j p_{ji} - \alpha_j q_{ji}) = g_i, \quad i=0,1,\ldots,m. \tag{9}$$

where p_{ji} and q_{ji} are the i^{th} samples of the j^{th} filters on the input and output, respectively, of the system as shown in Figure 2.2.

Figure 2.2 depicts a single-input, single-output linear system whose transfer function is denoted H^t and is referred to as the "true transfer function". The input to the system is denoted p^t and is not necessarily an impulse. The output to the system is denoted q^t. Both the input and output are corrupted due to measurement error to form the measured input, p^m, and the measured output, q^m. An estimate of the true transfer function is to be constructed. The poles and residues at the estimated transfer function are estimates of the resonances and residues of (1) for the case of nonimpulse excitation.

Equation (9) can be written in matrix form as $\Omega\ \theta = g$ where

$$\Omega = \begin{bmatrix} q_{00} & q_{10} & \cdots & q_{n0} & p_{00} & \cdots & p_{n0} \\ q_{01} & q_{11} & \cdots & q_{n1} & p_{01} & \cdots & p_{n1} \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ q_{0m} & q_{1m} & \cdots & q_{nm} & p_{0m} & \cdots & p_{nm} \end{bmatrix}$$

is the (M x 2N)-dimensional data matrix and $\theta \equiv [-\alpha_0, -\alpha_1, \ldots, -\alpha_n, \beta_0, \beta_1, \ldots \beta_n]^T$ is the 2N-dimensional parameter vector.

The filters in the model of Figure 2.2 are typically chosen as first-order discrete filters whose transfer functions, $F_j(z)$, may possess one zero or one pole or both. Once θ is adjusted to minimize g an estimate of the transfer function of the system can be formed as

$$H^m = \frac{\beta_0 F_0 + \ldots + \beta_n F_n}{\alpha_0 F_0 + \ldots + \alpha_n F_n} .$$

Estimates of the z-plane poles of the system, z_j' , may be taken as the poles of H^m. The A_j' may be taken as the residues of H^m.

The special case of Prony's method results from this general scheme when the $F_j(z)=z^j$ and the input is zero, i.e., $p_i^m=0$ for $i=0,\ldots,n$. In this case, $\Omega\theta=g$ reduces to $Qx=d$.

Any of the methods for constucting a parameter vector with Prony's difference equation can be applied, with suitable modifications, to construct θ. The most straightforward way to construct θ is to choose it to minimize $g^*g/\theta^*\theta$ which is equivalent to choosing θ as the weakest eigenvector of $\Omega * \Omega$ where the asterisk denotes the transpose conjugate.

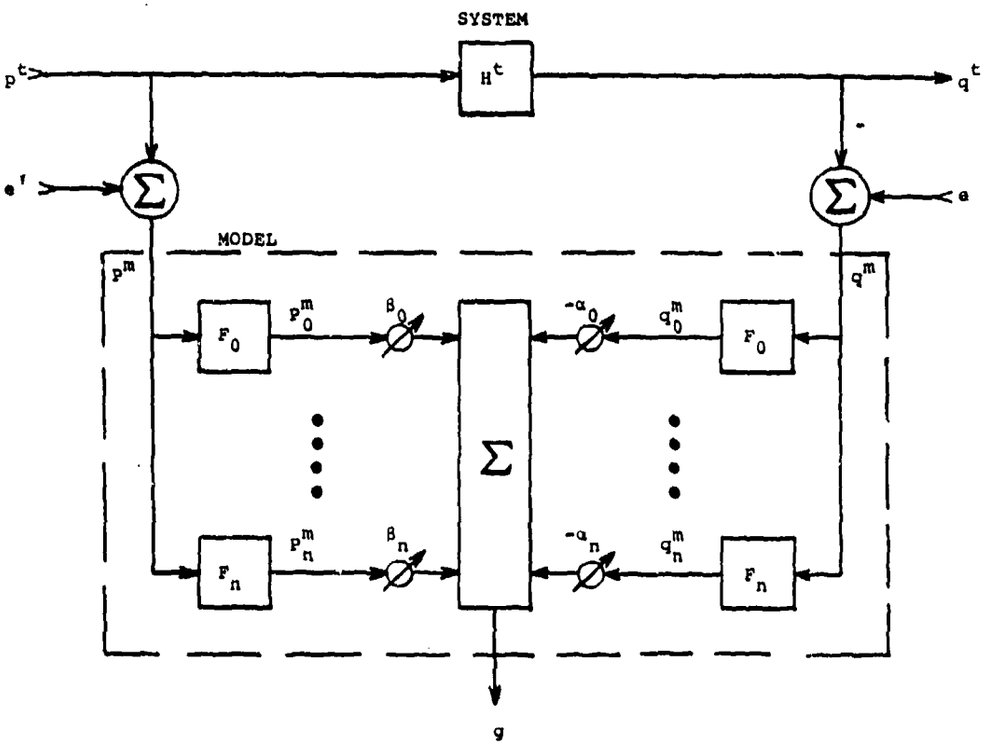

Figure 2.2. Generalized model of a linear system. e' and e are measurement error sequences.

2.12 SUMMARY AND CONCLUSIONS

Some of the more significant conclusions of this section are summarized below:

1. The analysis of the noise effects on the inhomogeneous solution and the lowest eigenvector of Q^TQ indicates that the lowest eigenvector is less sensitive to noise than the inhomogeneous solution. Hence, the inhomogeneous solution is considered less desirable of the two for resonance estimation.

2. All noniterative methods for finding a parameter vector; that is, the inhomogeneous solution, Jain's method, and EVD/SVD analysis; can be interpreted as methods of combining the eigenvectors of Q^TQ to construct a parameter vector.

3. Iterative techniques are fundamentally different from the noniterative techniques which we have examined in this section. Most iterative techniques choose the parameter vector to minimize the "true error" between the given data and an approximation to the data while noniterative techniques attempt to minimize "equation error". There is no reason to believe that minimizing "equation error" produces less accurate estimates than minimizing "true error". While it may be more satisfying to minimize "true error", it is questionable that the added expense and possibility of divergence associated with an iterative technique is merited by any gain in accuracy that might exist.

4. The iterative generalized lease-squares algorithm due to Clarke [21] is fundamentally different from the other iterative techniques reviewed in this section in that it does not attempt to minimize "true error". It is not completely clear what this method accomplishes since the criterion it attempts to satisfy, that is, "white residuals",

cannot be realized exactly with a finite data sequence. If we were to use an iterative technique for our application we would prefer that it satisfy some type of tangible criterion such as minimum "true error". For this reason the iterative generalized least-squares procedure is judged to be less useful that the other iterative techniques for our application.

5. All methods for constructing a parameter vector for Prony's difference equation carry over to the generalized model with suitable modifications where necessary.

6. For the purpose of automatic resonance extraction when it is known that n=k, the estimation procedure where the parameter vector is chosen as the weakest eigenvector of Q^TQ or $\Omega * \Omega$ is judged to be best of all iterative and noniterative methods. A similar procedure for the case where n>k is defined in the section on model order selection procedures.

3.0 PROCEDURES FOR ORDER SELECTION

3.1 INTRODUCTION

In Section 2, we assumed that k, the true number of resonances, which is equivalent to the rank of W, was known. In practical applications of resonance extraction procedures this assumption is unrealistic. The estimation of k for noisy data is not a simple problem since Q will be full rank or of rank N, whereas if there were no noise, Q = W, and the rank of W is easily determined.

Furthermore, in Section 2, we assumed that n = k. In this section we examine the case where n> k. Since the parameter vector is of length N = n + 1, there will be n resonance estimates produced by the estimation techniques of Section 2. There will then be n - k extraneous resonance estimates. We will present a procedure developed by Henderson [15] for solving this problem in this section.

In Section 2, it is shown that the noniterative procedures for estimating the resonances can be interpreted in terms of the EVD of Q^TQ. Each estimation method that we studied combined the eigenvectors in a particular way to form a parameter vector. It is not surprising that EVD analysis holds a central position with regard to all of these particular methods since EVD is simply a way of rearranging the information in a matrix in a form that reveals the essential characteristics of the information. Each estimation method is a particular way of combining the information to form resonance estimates. For this reason we adopt EVD analysis as a standard tool to be applied in any noniterative automatic resonance extraction procedure. Iterative procedures are not considered for this application because of their inherent problems such as lack of convergence and computational expense.

When $n = k$ we concluded in Section 2 that the weakest eigenvector of Q^TQ was the most "natural" approximation to the null eigenvector of W^TW where the term "null eigenvector" denotes an eigenvector corresponding to a zero eigenvalue. But in the case where $n > k$, there are $n - k$ null eigenvectors of W^TW. Are we then to take some combination of the $n - k$ weakest eigenvectors of Q^TQ as the most "natural" approximation to the "true" parameter vector? This question is addressed in this section.

Distributed linear systems such as a scatterer often have an infinite number of modes which means that k is infinite. What courses of action are available to us in this case? In practical cases there will be only a finite number of strong modes present in the data due to the fact that in any practical case the dominant frequencies present in the excitation will be limited to a frequency band of finite width. Highly damped modes will also be weak. The weaker modes will be lost in the measurement noise so that even though k may be, in truth, infinite, practical considerations dictate that k be estimated at a finite value.

This section concentrates on answering three questions:

1. How can the true number of resonances be estimated when $n > k$, k is unknown, and the noise level is unknown? (Order selection procedures)
2. How can the eigenvectors of Q^TQ be best combined to form a parameter vector once an estimate of the true order is available? (Procedures for constructing resonance estimates)
3. How can the extraneous resonance estimates be eliminated?

Only the simple case where Prony's difference equation can be applied is considered in this section.

3.2 STATISTICS OF THE EIGENVALUES OF Q^TQ

The problem of estimating the order or rank of W^TW reduces to the problem of discriminating which eigenvalues of Q^TQ correspond to zero eigenvalues of W^TW. If we assume that $M \geq N$ then Q^TQ will have no zero eigenvalues due to the noise in the data. In order to isolate the eigenvalues that would normally be zero we must come to some understanding of how the noise perturbs the eigenvalues. The first step toward this understanding is realizing that the eigenvalues of Q^TQ can be described as random variables. The statistics of these random variables depends to a large degree on the statistics of the noice in the data. In most practical circumstances the statistics of the noise will remain unknown. Beyond this, it is simply convenient to assume, in order to remain completely general, that the noise is altogether unknown but that it has certain very general characteristics by which it can be distinguished as noise. These characteristics are taken to be:

1. Uncorrelated from sample to sample or white. (If we assume that the noise is due to a very large or infinite number of very small uncontrolled influences which are sufficiently distant from each other to remain independent then the noise can be expected to be white).

2. Normally-distributed with zero mean and standard deviation of σ. (By the Central Limit Theorem [22] if a random variable is the sum of an infinite number of independent, zero-mean random variables, each with any distribution whatsoever, subject to certain very general constraints, then that random variable is normally-distributed and zero-mean.)

The assumption of white noise is believed to be the best assumption that can be made when nothing is known about the noise. This conclusion

is derived from the following line of logic: If the noise is correlated or nonwhite, it can be decomposed into a deterministic component and a purely random or uncorrelated component. The deterministic component of the correlated noise represents undesired information which cannot be distinguished from the desired information. The deterministic component must then be modeled by increasing the order of the difference equation model beyond that required by the desired information. The unmodeled portion of the noise is then uncorrelated and comprises the residuals of the model. An uncorrelated noise sequence represents pure randomness or lack of information and is the most difficult of all sequences to model. Summarizing this argument, if we are given only the data and absolutely no other information, we must assume that all deterministic components in the data contain information and the residual, purely random component is white noise. We desire to develop a procedure to estimate the "true" order based on the assumption that the noise corrupting the "true" data is white.

Given these assumptions about the statistics of the noise, a description of the statistics of the eigenvalues is desired. The statistical literature falls considerably short of an adequate analytical description of the statistics of the eigenvalues of a matrix such as Q^TQ although crude approximations to the statistics can be found [23]. Derivation of an analytic expression for the statistics is not a simple task. Our approach is to derive an adequate approximation for the eigenvalue statistics based on empirical studies. (An example of the type of studies performed is shown in Figure 3.2.)

Before defining a model of the eigenvalue statistics (or eigenstatistics for short) it is useful to examine the eigenvalues of the expectation or average value of Q^TQ. If the EVD of $W^TW = V^W\Lambda_N^W V^{WT}$, then the EVD of $\xi[Q^TQ] = V^W[\Lambda_N^W + LM\sigma^2]V^{WT}$. In other words the EVD of the expectation of Q^TQ is equivalent to the EVD of W^TW except that each eigenvalue has $M\sigma^2$ added to it. This is illustrated in Figure 3.1. The observed eigenvalues in specific random picks of Q have some distribution about these expected values. We desire to find an approximation for this distribution.

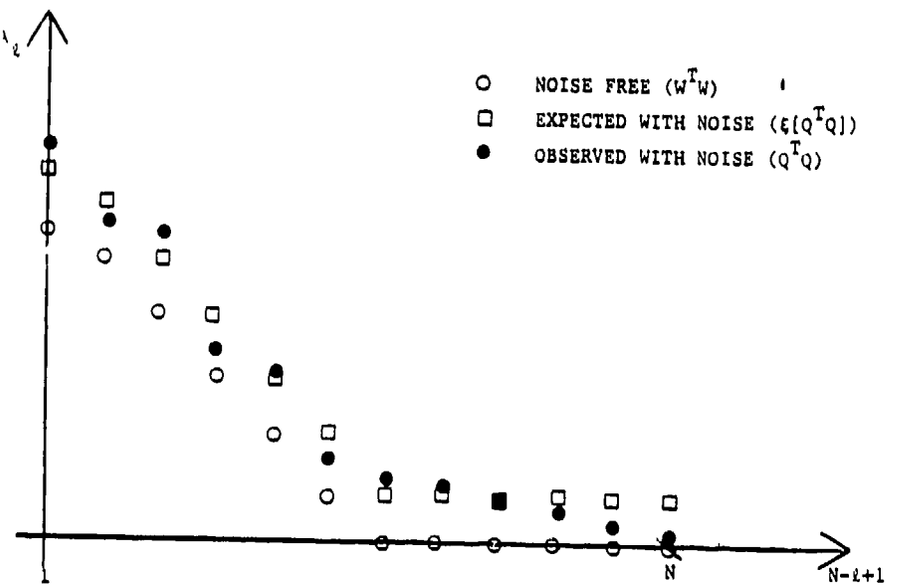

Figure 3.1. Illustration of eigenvalue statistics

The eigenvalue matrix has the form

$$\Lambda_N^W = \begin{bmatrix} \lambda_1^W & & & 0 \\ & \lambda_2^W & & \\ & & \ddots & \\ 0 & & & \lambda_N^W \end{bmatrix}$$

where $\lambda_1^W \leq \lambda_2^W \leq \ldots \leq \lambda_N^W$ are the eigenvalues of W^TW. The model that is proposed for the eigenstatistics can be written as a particular form of the EVD of Q^TQ:

$$Q^TQ = V[\Lambda_N^W + Y]V^T$$

where V is an (N x N)-dimensional orthogonal matrix whose columns are the eigenvectors of Q^TQ, Y is an (N x N)-dimensional diagonal matrix whose diagonal elements are samples of the random variable y. The frequency function for y is denoted f(y) and completely describes the eigenstatistics of the model. This model is chosen because it provides an adequate description of the statistics that have been observed in empirical studies.

The frequency function, f(y), can be approximated as

$$f_1(y) = K\left[\frac{4By}{(1 + By)^2}\right]^{2M/N}$$

where K is a scalar constant chosen so that

$$\int_0^\infty f(y)dy = 1,$$

and

$$B \equiv \frac{1}{(M - \frac{4}{3}N)\sigma^2} \quad .$$

The form of this frequency function is based on the frequency function found on page 261 of reference [23]. Substantial modifications were required to fit that frequency to empirically observed statistics.

This approximation or rule for the eigenstatistics is quite accurate when $M/N > 2$, is acceptable in accuracy when $3/2 < M/N < 2$, and is generally increasingly unacceptable in accuracy as M/N approaches 4/3 or falls below 4/3. In practical situations quite often $M >> N$ so that the rule usually is sufficiently accurate. Figure 3.2 compares this rule against the observed statistics for a particular value of M/N. The histogram of Figure 3.2 is formed from the results of 200 Monte Carlo trials where each trial consists of the EVD of Q^TQ and Q is composed entirely of noise samples. The noise was uncorrelated and normally-distributed with zero-mean and standard deviation σ. All thirteen eigenvalues from each trial were used to compile the histogram so that the histogram represents a compilation of 2600 eigenvalues. Since Q is composed entirely of noise, the eigenvalues of W^TW are all zero in this case or $\Lambda_N^W = 0$. It follows that the eigenvalues of Q^TQ by our model consist solely of samples of the random variable y. The approximation to the frequency function is plotted with the histogram for comparison in Figure 3.2. The rule has been tested at various other values of M and N with good comparisons in each case provided that $M/N > 3/2$. For the case shown we note that the rule seems to describe all features of the eigenstatistics.

The model of eigenstatistics provides information about how the eigenvalues of Q^TQ should behave. In particular, the model can be used to distinguish those eigenvalues of Q^TQ that correspond to zero eigenvalues of W^TW since these eigenvalues consist entirely of samples of the random variable y and no other component, whereas, the other eigenvalues have deterministic components that serve to distinguish them. The method in which the model is applied to discriminate between the eigenvalues is covered in our description of order selection procedures which follows.

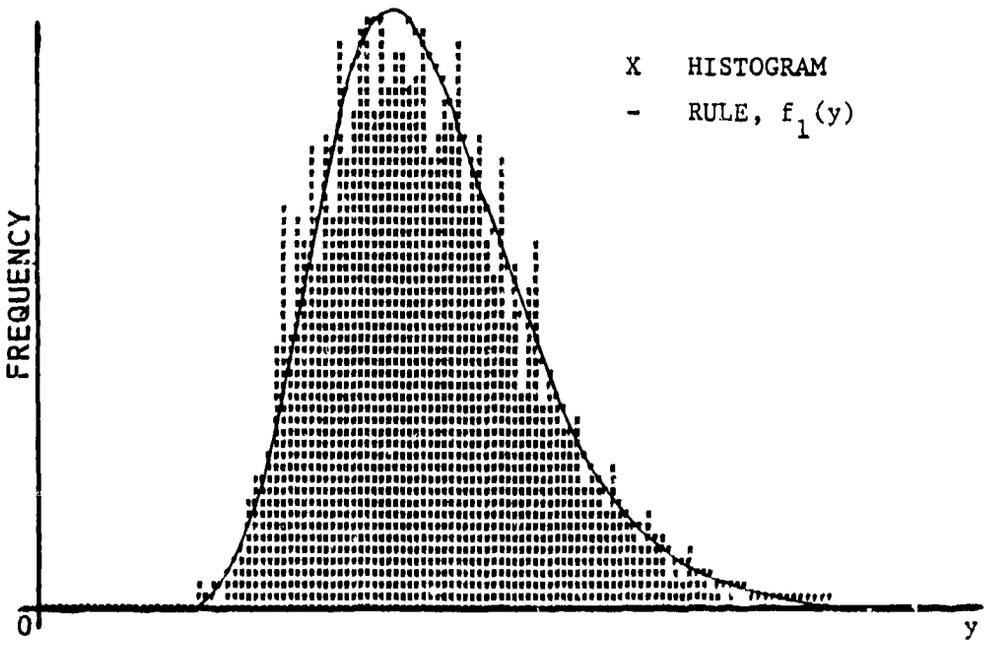

Figure 3.2. Comparison between observed eigenstatistics and rule.

M/N = 200/13.

3.3 PROCEDURES FOR ESTIMATING THE TRUE ORDER

Here we consider the problem of estimating k, the "true" number of resonances given the EVD of Q^TQ for some value of $n > k$. Inaccurate resonance estimates result when n is less than the number of dominant resonances since in this case the unmodeled resonances appear as a large noise component. It must be assumed that n can be chosen sufficiently large for the resonance extraction procedures to be effective.

In practical cases the noise level is unknown. The order selection procedure we use then should not depend on this information. We have found three procedures that are suited for this problem:

1. The likelihood ratio criterion [24].
2. Akaike's Information Criterion [24,25].
3. Maximum likelihood criterion developed by ETI.

The first two criteria were designed for use with the minimum quadratic error for a particular model order. Use of these criteria required that a model be fitted to the data for all orders ℓ of possible interest. Because fitting a model at large number of possible values for ℓ is an expensive procedure and because we desire to use the EVD to determine the model order, we adapted the first two criteria to use the eigenvalues of Q^TQ instead of the QE. What effect this modification has on the results will be discussed when each criteria is described.

The likelihood ratio criterion after being adapted for the use with the eigenvalues, can be stated simply as

$$1 - \frac{\lambda_\ell}{\lambda_{\ell+1}} < \delta$$

$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. This criterion is based on the principle that the eigenvalues corresponding to zero eigenvalues of W^TW will be nearly equal, especially when $M/N >> 1$. δ is the threshold below which the eigenvalues are considered

equal. The order is chosen somewhere in the vicinity of where the criterion is satisfied for several consecutive values of ℓ. The criterion will in general not continuously increase as ℓ increases because the random nature of the eigenvalues is such that the test must be successful several times to be sure that the proper order has been reached. Use of the criterion is illustrated in Figure 3.3. The order is chosen as $k' = N - \ell'$, where ℓ' is the value of ℓ for which the criterion is satisfied. The δ parameter could be chosen by use of the eigenvalues model developed in this section.

Akaike's criterion as it has been adapted for use with the eigenvalues is

$$AIC(\ell) = \ell n\ \lambda_{N-\ell+1} + \frac{2\ell}{M + N - 1}$$

The criterion is evaluated for $\ell = 1,\ldots,N$. If the value of ℓ that minimized AIC is denoted ℓ', the estimated order is then $k' = N - \ell'$. In order to adapt this criterion for our purposes, we have probably taken a few more liberties than Akaike would have permitted. If the criterion defined above does not work well it should not be taken as a true reflection of the performance of the original criterion. Figure 3.4 illustrates the application of Akaike's criterion.

The third order selection procedure is the maximum likelihood (ML) criterion. This criterion was developed specifically for application to the eigenvalues of Q^TQ. This procedure examines the smallest ℓ eigenvalues and assesses the likelihood that those eigenvalues are due to noise only or, in other words, correspond to the zero eigenvalues of W^TW. This is done by comparing the observed eigenvalues to the known frequency function of y.

The standard measure of how well a set of observed values match a known distribution is the likelihood function. The likelihood function that is used for this criterion is the mean log-likelihood defined as

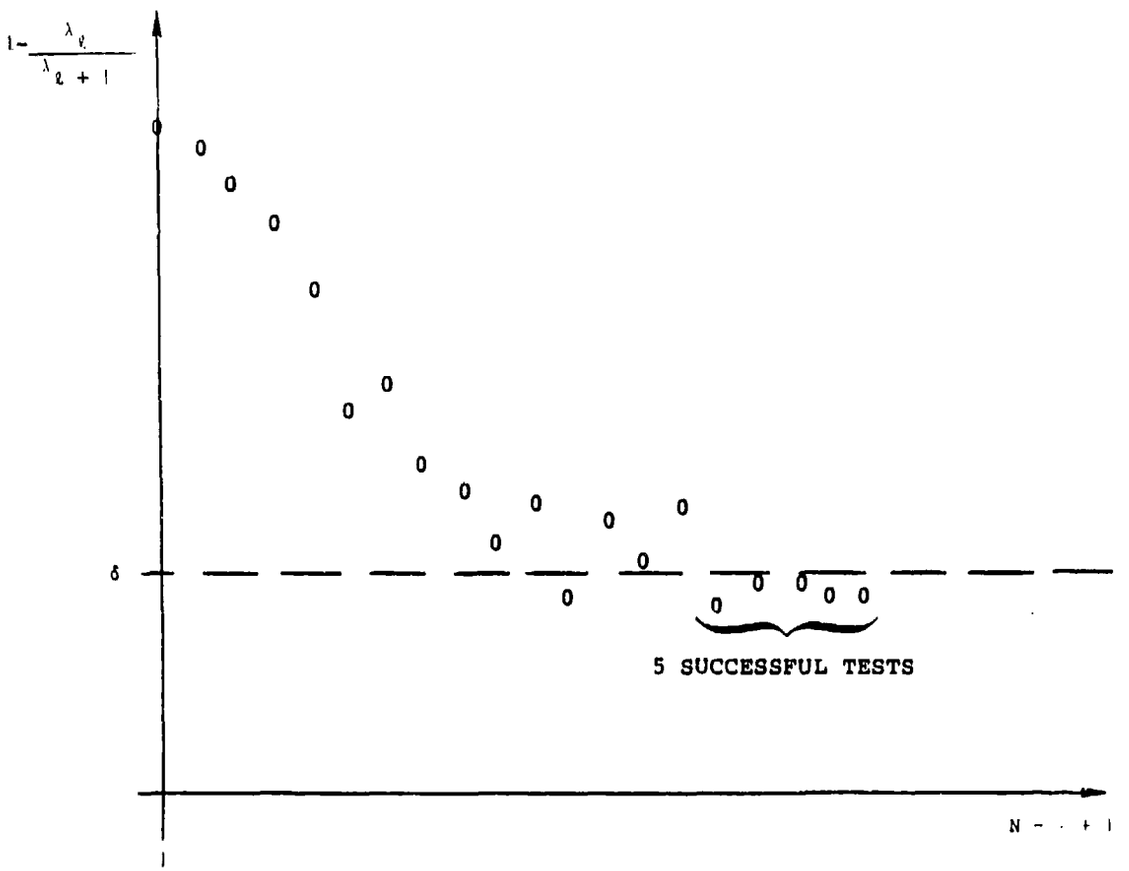

Figure 3.3. Illustration of likelihood ratio criterion

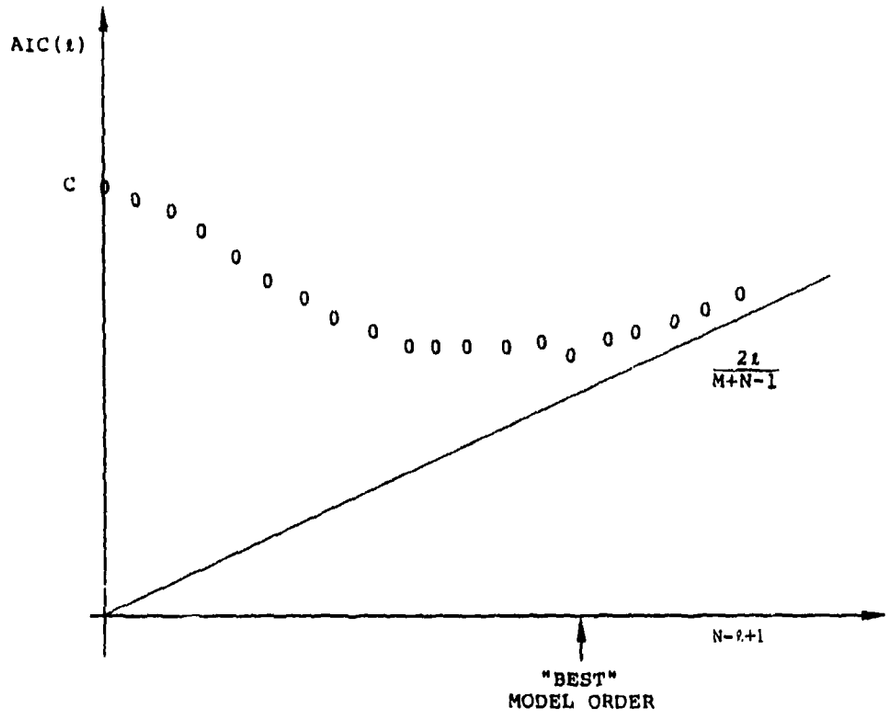

Figure 3.4. Illustration of Akaike's criterion

$$L_\ell\ (\lambda_1,\lambda_2,\ldots,\lambda_\ell;N,M,\sigma_{est}) = \frac{1}{\ell} \sum_{k=1}^{\ell} \ell n\ f_1\ (\lambda_k;N,M,\sigma_{est}) - L_o\ (N,M)$$

where L_o is the expected log-likelihood assuming the eigenvalues are due to noise only and is defined as

$$L_o\ (N,M) = \int_0^\infty f_1\ (y;N,M)\ \ell n\ f_1\ (y;N,M)dy$$

The L_o term normalizes the criterion relative to the expected likelihood. σ_{est} is an adjustable parameter corresponding to the unknown standard deviation of the noise, σ; and

$$f_1\ (\lambda_k;N,M;\sigma_{est}) = K \left[\frac{4B\lambda_k}{(1+B\lambda_k)^2}\right]^{2M/N}$$

$$\text{where } B \equiv \frac{1}{(M - \frac{4}{3}N)\sigma_{est}^2}$$

Because the standard deviation of the noise is unknown, σ_{est} must be adjusted to maximize L_ℓ. The maximum of L_ℓ is taken to be the likelihood that the ℓ smallest eigenvalues are due to noise. The value of σ_{est} that maximizes L_ℓ is an extimate of the unknown noise level. When the maximum of L_ℓ falls below a certain value, which is called the cut-off likelihood and is denoted L_c, the observed eigenvalues are so widely distributed that the extreme values fall well into the tails of the frequency function regardless of how σ_{est} is adjusted. Presumably when this occurs some of the eigenvalues are not due solely to noise and deviate significantly from the noise eigenvalues.

The criterion can be interpreted in the following way. σ_{est} is adjusted to fit the set of observed eigenvalues into the statistical distribution for noise eigenvalues. When the best fit possible does not yield a sufficiently high likelihood that the eigenvalues are due only to noise then the criterion is triggered.

The maximum likelihood procedure for order selection consists of the following steps:

1. For $\ell = 2,3,\ldots,N$ adjust σ_{est} to maximize L_ℓ.
2. Choose ℓ' as the largest value of ℓ such that

$$\max_{\sigma_{est}} L_i > L_c \text{ for } i = 2,3,\ldots,\ell.$$

3. The selected order is then $N - \ell'$ and the value of σ_{est} that maximizes $L_{\ell'}$ is an estimate of the noise level, σ.
4. If $\max_{\sigma_{est}} L_\ell > L_c$ for $\ell = 2,\ldots,N$, then the procedure cannot discern a significant difference between the observed eigenvalues and the eigenvalues of a matrix with purely random data.

The choice of the cut-off likelihood, L_c, determines how sure one wants to be that the lower eigenvalues are due to something other than noise. When the observed eigenvalues are due to noise only, the maximum of L_ℓ tends to vary randomly about zero. Figure 3.5 shows a histogram of the maximum of L_ℓ. From the histogram it can be concluded that if L_c is chosen as -1 then there is approximately one chance in a hundred that the criterion will be triggered due to noise only. Because of the nature of the likelihood function, this false alarm rate for $L_c = -1$ should be relatively indifferent to the values of M, N, or ℓ. $L_c = -1$ should therefore represent a false alarm rate of approximately one in a hundred for all problems.

3.4 PROCEDURES FOR CONSTRUCTING A PARAMETER VECTOR

Now that several possible procedures for selecting the order have been defined, we consider the questions:

1. How can the eigenvectors of Q^TQ be best combined to form a parameter vector for resonance estimation?
2. How can the extraneous resonance estimates be eliminated?

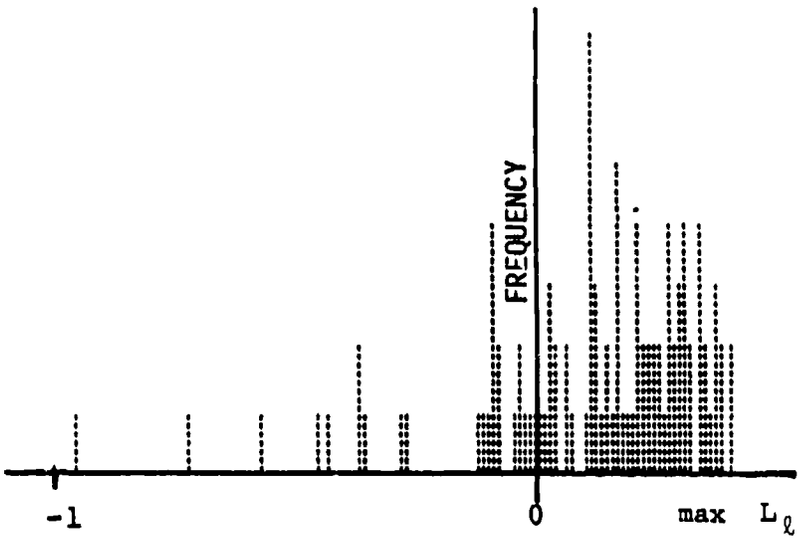

FIgure 3.5. Histogram of log-likelihood function for eigenvalues due to noise only. 100 Monte Carlo trials. M = 512, N = 16, and ℓ = 8

In Section 2, it is demonstrated that all the noniterative resonance estimation techniques, e.g., the inhomogeneous solution, Jain's method, or choosing the weakest eigenvector, can be interpreted as different ways of combining the eigenvectors of Q^TQ to form resonance estimates. We concluded that choosing the weakest eigenvector as the "best" estimate of the null eigenvector of W^TW was the most "natural" estimation procedure for the case when n = k. For the case when n > k the "natural" estimate of the parameter vector is less clearly defined.

Any vector, x^ℓ, in the space spanned by the ℓ weakest eigenvectors of Q^TQ satisfies

$$\lambda_\ell \geq \frac{(Qx^\ell)^T(Qx^\ell)}{(x^\ell)^T(x^\ell)}$$

where $N - \ell$ is the estimate of the "ture" order. Stated differently, any vector in the space spanned by the ℓ weakest eigenvectors produces a value of NQE less than the value of λ_ℓ. It follows that any x^ℓ can be consiuered as an approximate solution to Prony's difference equation. Unfortunately, there are many choices for x^ℓ. In addition, x^ℓ is N-dimensional so that it will yield n resonances when $N - \ell$ is the appropriate number of resonances. Henderson [15] devised a method to resolve these difficulties.

Henderson's method is based on a concept he termed "the ℓ^{th} autoregression nullspace" which is defined as the rowspace (the space spanned by the rows) of the $(\ell \times N)$-dimensional matrix

$$G_\ell = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_k & 0 & 0 & \cdots & 0 \\ 0 & \alpha_0 & \cdots & & \alpha_k & 0 & \cdots & 0 \\ & \vdots & & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_k \end{bmatrix}$$

where the α_i are, in this case, the elements of the true parameter vector, the true order is k, and $N = k + \ell$. Let $h = [h_0, h_1, h_2, \ldots, h_n]^T$ be any vector in the row space of G_ℓ. Henderson proved that the roots of the polynomial,

$$\sum_{j=0}^{n} h_j z^j = 0, \tag{10}$$

constitute a superset of the roots of the "true" polynomial,

$$\sum_{j=0}^{k} a_j z^j = 0.$$

The roots of the true polynomial are the exact z-plane poles of the system. The other ℓ-1 roots of (10) represent extraneous roots. A vector h can be found to produce any specified set of extraneous roots.

The essence of Henderson's method consists of approximating the rowspace of G_ℓ by the space spanned by the ℓ' weakest eigenvectors of Q^TQ. By making this approximation it is possible to form an estimate of the "reduced" $(k'+1)$-dimensional parameter vector. Here $N = k'+\ell'$ where k' is an estimate of the true order k. The procedure is:

1. Form a matrix whose rows consist of the ℓ' weakest eigenvectors of Q^TQ.
2. Use a Gaussian elimination procedure on the matrix of step 1 to form an $(\ell' \times N)$-dimensional matrix with zeros in the opposing corners. The matrix has the form:

$$\begin{bmatrix} XXXX0000 \\ 0XXXX000 \\ 00XXXX00 \\ 000XXXX0 \\ 0000XXXX \end{bmatrix}$$

   where "X" denotes a nonzero element. This matrix is intended to approximate G_ℓ in some sense.
3. Form an $(\ell' \times k' + 1)$-dimensional matrix, H , by eliminating the zeros and appropriately shifting the rows of the matrix in step 2. Each row of this matrix should be an approximation to the true parameter vector in some sense.

4. The "best" estimate of the true parameter vector is the strongest eigenvector of $H^T H$.

This procedure is an ingenious way of getting around the difficulties we mentioned previously and is intuitively pleasing. However, if we choose to be critical, it is clear that some very perplexing questions could be asked about this procedure. For example, what error criterion or other quality measure of the estimates does the reduced parameter vector obtained with this procedure satisfy?

On further study of this procedure, other unanswered questions arise:

1. The procedure begins by loading the eigenvectors into a matrix. Perhaps the eigenvectors should be weighted somehow according to their relative merits in estimating the resonances before being loaded into the matrix. The weights might be based on how well the corresponding eigenvalues fit within distribution for the noise eigenvalues in the third step of the ML procedure or on the values of the corresponding eigenvalues themselves. How should the weights be defined?

2. The use of Gaussian elimination to form an estimate of G_ℓ may not be the best procedure to use since Gaussian elimination can emphasize certain directions in the space spanned by the eigenvectors that were not actually dominant originally. This emphasis of certain directions could unduly perturb the reduced parameter vector. It may be possible to use certain orthogonal transformations in place of Gaussian elimination so that the magnitudes associated with the directions are not altered. What specific procedures can be applied in place of Gaussian elimination?

These questions merit further study.

3.5 NUMERICAL EXAMPLES

The waveform used in the numerical example of Section 2 is used again for these numerical examples which illustrate the order selection procedures and the procedure to eliminate extraneous resonances. The waveform is corrupted with uncorrelated, normally-distributed noise with zero-mean and standard deviation $\sigma = 0.01$. The data were used to fill the (M x N)-dimensional matrix Q where M and N were chosen to be 379 and 20, respectively. The number of samples used in filling Q is $M + N - 1 = 398$. The last two samples of the waveform were unused. The eigenvalues resulting from the EVD of Q^TQ are shown in Figure 3.6. We note that it is readily apparent to the eye which eigenvalues are due solely to noise.

Table 3.1 displays the resonance estimates obtained using the weakest eigenvector of Q^TQ and by using Henderson's procedure to eliminate the extraneous poles. Since the poles must occur in conjugate pairs when they are complex, we only show the poles in the upper half of the s-plane (positive-imaginary region of s-plane). The extraneous resonances that result from the weakest eigenvector are not shown. Henderson's algorithm does, in fact, eliminate the extraneous resonances and does yield relatively accurate estimates of the true poles. But when the estimates using Henderson's procedure are compared to those using the weakest eigenvector, we note that Henderson's procedure produces less accurate estimates. The reasons behind the less accurate results are thought to be related to some of the unanswered questions about Henderson's procedure which are enumerated in this section.

Table 3.2 displays the results of applying the maximum likelihood order selection procedure to the observed eigenvalues in Figure 3.6. We recall that this procedure selects an integer, ℓ', as the largest value of ℓ such that

$$\max_{\sigma_{est}} L_i < L_c \text{ for } i = 2,3,\ldots\ell \ .$$

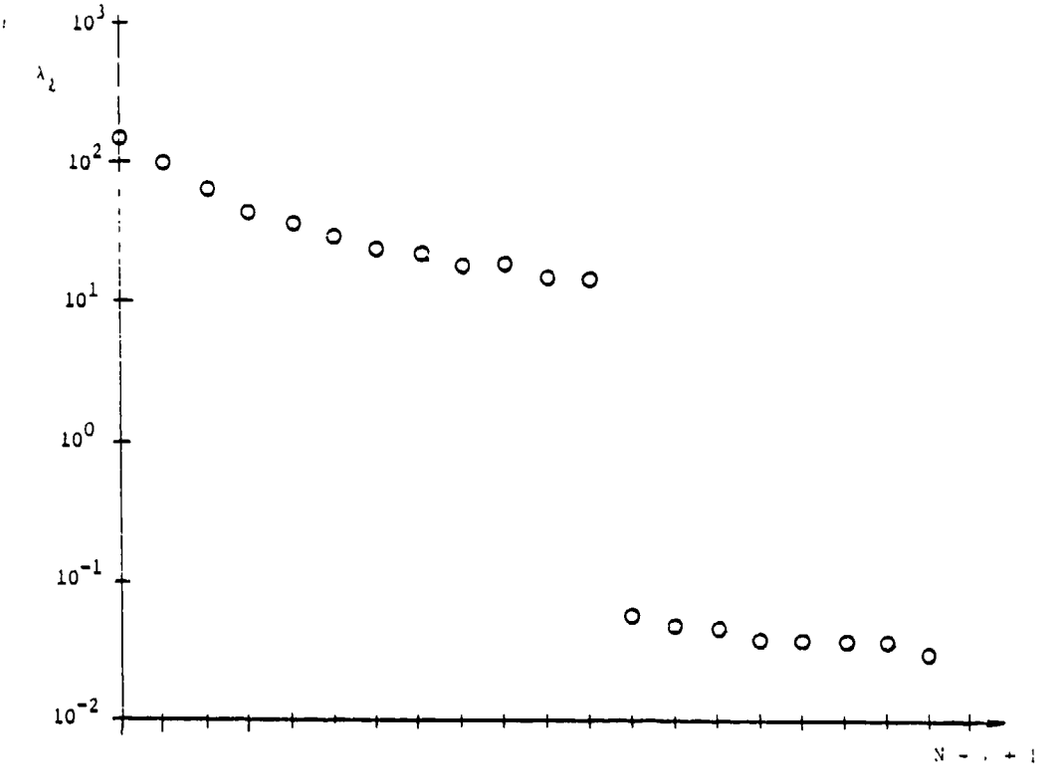

Figure 3.6. Eigenvalues of Q^TQ for numerical example.

Table 3.1. Comparison of resonance estimates in terms of s-plane poles.

Real Parts of Poles

True	Estimated	
	Full ** Order	True* Order
-0.082	-0.082	-0.128
-0.147	-0.146	-0.118
-0.188	-0.185	-0.100
-0.220	-0.218	-0.232
-0.247	-0.251	-0.299
-0.270	-0.269	-0.276

Imaginary Parts of Poles

True	Estimated	
	Full ** Order	True* Order
0.926	0.926	0.955
2.874	2.873	2.965
4.835	4.838	4.819
6.800	6.802	6.722
8.767	8.766	8.743
10.733	10.737	10.746

* Estimates for true order formed by using Henderson's procedure.

** Estimates using the weakest eigenvector of Q^TQ. The extraneous poles are not shown.

Table 3.2. Results of application of the Maximum Likelihood order selection procedure.

ℓ	max L_ℓ	σ_{est}
2	0.435	0.009778
3	0.431	0.009991
4	0.439	0.01011
5	0.444	0.01021
6	0.355	0.01045
7	0.305	0.01065
8	0.191	0.01096
9	-93.9	------
10	-93.9	------
11	-93.9	------
12	-93.9	------
13	-93.9	------
14	-93.9	------
15	-93.9	------
16	-93.9	------
17	-93.9	------
18	-93.9	------
19	-93.9	------
20	-93.9	------

The column in the table labeled max L_ℓ tabulates the maximum of L_ℓ obtained by adjusting σ_{est} for each value of ℓ. If the cut-off likelihood were chosen as $L_c = -1$ (which is the recommended value), then the procedure selects $\ell' = 8$ since this value satisfies the above requirements. The selected order is then $N - \ell' = 12$ which is exactly the true order. We conclude that this procedure works very well for this particular example. A more severe test can be imagined, however, where the two groups of eigenvalues are not so clearly distinguishable. Also, note that the method yields a very reasonable estimate of the noise level at $\ell = 8$.

An attempt was made to perform a more severe test but representative results were not produced. The crude algorithm we are currently using to search for the maximum likelihood breaks down for the more severe case. The problem, though, is purely mechanical and does not indicate a true limitation of the method. The true test of the procedure is its application to real, measured data.

Table 3.3 shows the results of applying Akaike's criterion to the observed eigenvalues. The minimum value of AIC (ℓ) defines the order. From Table 3.3, we observe that the minimum occurs at $\ell = 1$. One wonders if a larger value of N were used whether AIC(ℓ) would go through a clearly defined minimum. If we assume that $\ell' = 1$ is an accurate prediction, then the selected order is $N - \ell' = 19$ which is not the true order. We conclude that Akaike's criterion does not work very well when applied to the eigenvalues. This conclusion does not indicate that Akaike's original criterion is not effective since the criterion we have used has been considerably modified to enable application to the eigenvalues. Akaike's original criterion was never intended to be used in this way.

Table 3.4 displays the results of applying the likelihood ratio criterion to the observed eigenvalues. We observe that the values of $1 - \lambda_\ell/\lambda_{\ell+1}$ for $\ell < 8$ do not differ significantly from the values for $\ell > 8$. There is then no clean break point for choosing the order. We conclude that the likelihood ratio criterion, as we have applied it, is not suitable for selecting the order from the observed eigenvalues.

Table 3.3. Results of application of Akaike's criterion for order selection.

ℓ	AIC(ℓ)
1	-3.40
2	-3.21
3	-3.18
4	-3.18
5	-3.16
6	-2.96
7	-2.94
8	-2.80
9	+2.77
10	+2.79
11	+2.98
12	+3.00
13	+3.18
14	+3.26
15	+3.40
16	+3.60
17	+3.82
18	+4.18
19	+4.59
20	+4.97

Table 3.4. Results of application of the likelihood ratio criterion.

ℓ	$1 - \lambda_\ell / \lambda_{\ell+1}$
1	0.1812
2	0.0293
3	0.0102
4	0.0225
5	0.1844
6	0.0201
7	0.1399
8	0.9962
9	0.0257
10	0.1744
11	0.0213
12	0.1722
13	0.0829
14	0.1329
15	0.1900
16	0.1996
17	0.3057
18	0.3382
19	0.6802

3.6 SUMMARY AND CONCLUSIONS

Some of the more significant results of this section are summarized below.

1. Henderson's procedure for constructing a parameter of reduced dimension has been tested on synthetically-generated data and has produced accurate resonance estimates although the estimates were less accurate than those obtained using the weakest eigenvector procedure. Some unanswered questions about Henderson's procedure merit further study.

2. Three methods for estimating the true number of resonances were introduced in this section. On testing each procedure with synthetically-generated data we found that only one of the procedures gave a reasonable estimate of the true number of resonances. This method was the maximum likelihood procedure.

3. A model has been developed for the statistics of the eigenvalues of Q^TQ assuming that the noise corrupting the waveform is white and gaussian-distributed. The model is used in the ML procedure for selecting the order.

4. The HFTI algorithm [16] is judged to be less useful for order determination because it provides no means of constructing a parameter vector.

5. Another method for order estimation is the time-reversal technique described in Appendix I. Almost nothing is known about the precise way in which this procedure is able to discern which resonances are extraneous. We are, therefore, hesitant to recommend this procedure until it is better understood.

4.0 AUTOMATIC PROCESSING CONSIDERATIONS

4.1 THE AUTOMATIC RESONANCE EXTRACTION PROBLEM FOR RADAR TARGET IDENTIFICATION

There are two types of resonance extraction problems that can be associated with the radar target identification application:

1. Resonance extraction for target recognition.
2. Resonance extraction for target intelligence.

All that is required in the first problem is to select a pole set, from a library of resonances for known targets, that best characterizes an observed waveform. The observed waveform, in this case, is the time-domain radar return from the target of interest. This problem, depending on how it is approached, can be much simpler than the second problem. The second problem involves the identification and characterization of a previously unknown target. In contrast to the first problem, this problem has no apriori information to work from.

The first requirement in each of these problems is that the radar return be measured in some fashion. We discuss some preferred ways in which the measurements should be performed next.

4.2 THE DATA ACQUISITION PROBLEM

There are three parameters that describe the mode of measurement of a sampled transient waveform which are important for the effectiveness of resonance extraction procedures. These are the period of observation, T, the sampling rate, ω, and the number of samples measured, N_M. The period of observation is simply the time duration of the waveform from the first sample to the last. The sampling rate is defined by $\omega = 2\pi/\Delta t$ where Δt is the time interval between successive samples. These parameters are related by $T = 2\pi N_M/\omega$.

There are four more parameters that describe a "region of feasibility" and a "preferred locus" in ω - T space shown in Figure 4.1. These parameters are:

1. The expected period of time that the transient returns are above the noise, T_{max},
2. One-half cycle of the lowest frequency of interest, $T_{min} = \pi/\omega_{LOW}$,
3. The highest sampling rate available or feasible on the measurement apparatus, ω_{max}, and
4. The Nyquist sampling rate for the highest frequency of interest, $\omega_{min} = 2\omega_{HIGH}$.

If adequate measurements of the frequencies of interest are to be made, ω and T must fall within the region of feasibility. For adequate measurements to be possible at all, it is required that the region of feasibility exist or that $\omega_{max} \geq \omega_{min}$. Also, the possibility of adequate measurements requires that the maximum number of samples that is feasible, $(N_M)_{max}$, satisfy

$$(N_M)_{max} \geq \frac{\omega_{min} T_{min}}{2\pi}$$

The preferred locus defines the most desirable portions in the region of feasibility (and outside this region as well). It should be noticed that the locus is a broken line. The corners of the locus have special meanings. The corner at ω_{min} and T_{max} represents the point where holding the sampling rate constant while increasing the number of samples ceases to provide any more information. Any further increase in the period of observation improves nothing since the added portions of the waveform are dominated by noise. Further information can be provided only by increasing the sampling rate while increasing the number of samples. Thus the corner is formed. By continuing to increase the number of samples the second corner is reached. This corner represents the point where further increase in the sampling rate is impossible. At this point the only option available is to increase the period of observation (even though this provides little benefit) while increasing the number of samples.

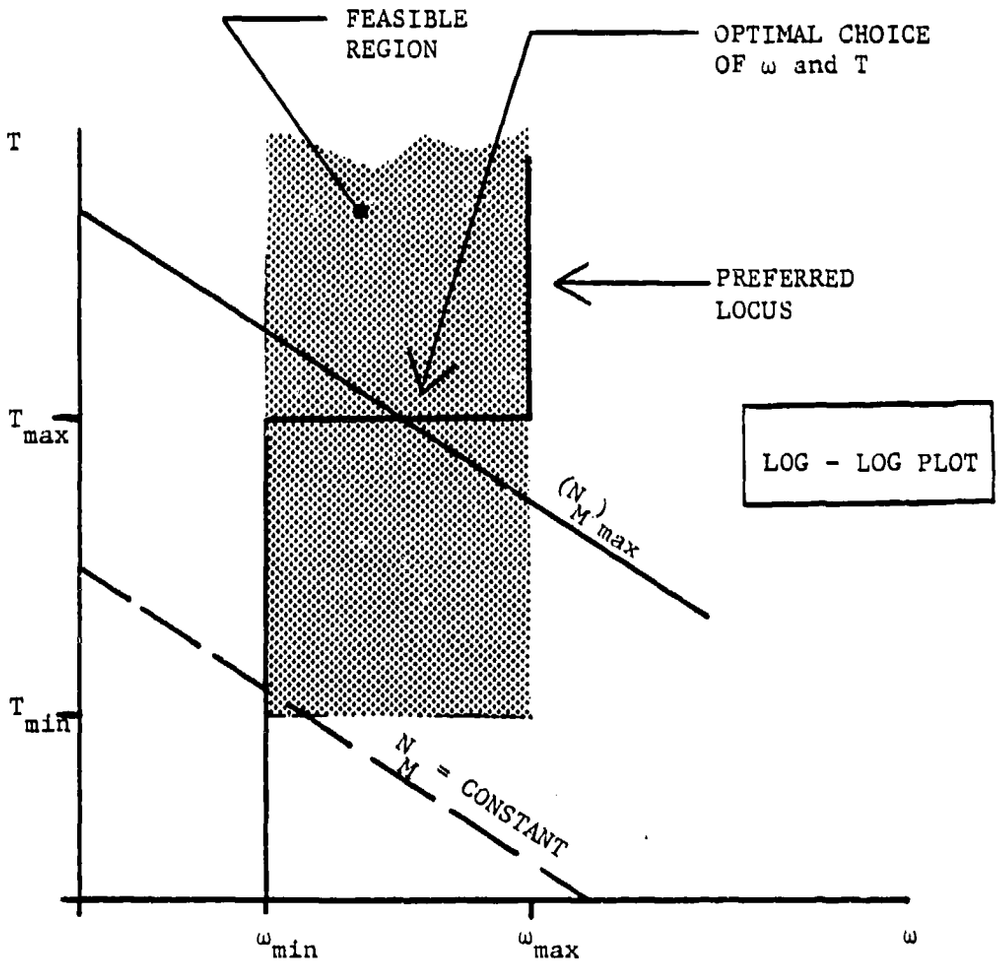

Figure 4.1. Guidelines for choosing sampling rate and period of observation for resonance extraction.

The primary objective in data acquisition is compiling the maximum amount of information about the object of interest. This can be done best if the largest number of samples that is feasible to measure and store on the measurement apparatus is used. Therefore, the point on the preferred locus for which $N_M = (N_M)_{max}$ defines the best values of ω and T for resonance extraction.

We realize that there is much more to the data acquisition problem than simply selecting the sampling rate and period of observation but beyond just making accurate measurements these are the only parameters of concern for resonance extraction.

4.3 TWO FACTORS EFFECTING THE ACCURACY OF PRONY'S METHOD

Quite often in the literature the sensitivity of Prony's method to noise in the data has been noted. The sources of this sensitivity can be isolated to two factors: dense sampling and bias. The term "dense sampling" means that the sampling rate of the waveform is much higher than the sampling rate required to faithfully record the highest frequency of interest or the Nyquist frequency. The dense sampling problem can be ameliorated by proper manipulation of the data. The bias problem can also be ameliorated and is discussed after the dense sampling problem.

4.3.1 THE DENSE SAMPLING PROBLEM

Kulp [14] has performed an excellent study of the effects of the sampling rate on the accuracy of the estimates obtained by Prony's method. The effect which Kulp demonstrated can be explained by the following simple

example. Consider the noise-free exponential waveform, $w = \exp(-\alpha t)$. Two samples of this waveform, which are spaced Δt apart in time, are given by $w_0 = \exp(-\alpha t)$ and $w_1 = \exp(-\alpha t - \alpha \Delta t)$. The "resonance" for this waveform is an s-plane pole at $s = -\alpha$. The two noise corrupted samples q_0 and q_1 which correspond to w_0 and w_1 can be used to form an estimate, s_1, of the single resonance as

$$s_1 = \frac{1}{\Delta t} \ln \frac{q_1}{q_0} \quad .$$

This expression is nothing more than a first-order Prony procedure to estimate the single resonance. It can be verified that

$$-\alpha = \frac{1}{\Delta t} \ln \frac{w_1}{w_0} \quad .$$

The point to be made is that the error in the estimate, $s_1 + \alpha$, is inversely proportional to the sampling interval

$$s_1 + \alpha = \frac{1}{\Delta t} \ln \frac{q_1 w_0}{q_0 w_1} \quad .$$

This same point can be made for higher orders and for the least-squares version of Prony's method as well as all procedures that are based on Prony's difference equation. Another quantity that displays a dependence on the sampling interval is the condition number of the matrix $\bar{W}^T\bar{W}$ (in the notation of this report) which is used in the least-squares version of the Prony procedure. In fact, Kulp used the condition number to derive bounds on the error in the coefficients of Prony's difference equation. The dependence of the condition number and the dependence of the error in this example on sampling rate are different aspects of the same phenomenon. By this example we have briefly explained the problem of dense sampling.

When we are given a waveform with no other information how do we know whether it is densely sampled or not? One possible procedure is to take the Fourier transform of the waveform to determine its frequency content. If most of the energy is concentrated in the lower frequencies, the waveform is densely sampled.

One way of removing the effect of dense sampling is to use a pre-processing technique such as the redundant-averaging scheme described in Appendix E of this report. Another, perhaps more direct, way is altering the way in which the data matrix is filled. For example, if the Fourier transform indicates that the sampling interval can be increased D times, where D is an integer, without aliasing any of the dominant frequencies of the signal, then an alternate form of the data matrix can be used as before to estimate the resonances:

$$Q_A = \begin{bmatrix} q_0 & q_D & q_{2D} & \cdots & q_{nD} \\ q_1 & q_{D+1} & q_{2D+1} & \cdots & q_{nD+1} \\ \vdots & \vdots & \vdots & & \vdots \\ q_m & q_{D+m} & q_{2D+m} & \cdots & q_{nD+m} \end{bmatrix}$$

where $m \geq D - 1$ is required if all of the data are to be used. In order to obtain the proper s-plane pole estimates Δt must be multiplied by D.

4.3.2 THE BIAS PROBLEM

The bias problem is very complex. There are different definitions for the term "bias". The definition that we use is the standard definition of bias found in basic textbooks of statistics [22]. If we denote the estimated parameter vector as x and the true parameter vector as $\bar{x}$ then x is said to be an unbiased estimate if

$$\xi[x] = \bar{x}$$

where x is an N-dimensional random variable whose random nature is due to the randomness of the noise in the data. The unbiased nature of any estimate is strongly dependent on the particular procedure used to form the estimate.

The bias problem can be observed and approached from many directions. Eykhoff [20] describes the "asympotic bias" in terms of "correlated residuals". The bias can be described in terms of suboptimal parameters in the sense that they do not minimize the "true error" defined by (5) although this approach to the problem involves factors other than just the bias such as the choice of error criterion. The fact that equation error of Prony's difference equation is minimized by the inhomogeneous solution might lead one to conclude that the particular choice of equation error as the quantity to be minimized causes the bias. On closer examination we must disagree with this conclusion. It is not the choice of error criterion that causes the bias but how the error criterion is minimized or, equivalently, how the minimizing parameter vector is constructed. Our observations of Section 2 seem to indicate that if the parameter vector is chosen as the weakest eigenvector of the transpose product of the data matrix, then the best estimates are obtained for the case when n = k. The symmetrical perturbations expected in the eigenvectors with uncorrelated noise seem to indicate that the expectation of the weakest eigenvector is equivalent to its "true" value without noise so that the weakest eigenvector should be unbiased. At the present time, we have no rigorous proof that the weakest eigenvector is unbiased but we feel that it is capable of proof.

If the noise is Gaussian-distributed and uncorrelated, using an iterative least-squares technique to minimize "true error" produces an

unbiased estimate. Unfortunately, iterative techniques are not well-suited for our application.

For the case when $n > k$, Henderson's procedure (Section 3) should effectively avoid the problems of bias based on our current understanding of the bias problem.

One method of treating both the bias problem and the dense sampling problem while still using the inhomogeneous solution is to simply increase the number of poles modeled or n. This method is described in Appendix B. Unfortunately, this method suffers from the problem of extraneous poles.

We believe that the previous so-called problems associated with Prony's method stem from the failure to cure both the bias problem and the dense sampling problem. Unfortunately, curing one problem does not cure the other so that it is very easy to fall into one trap, if not both. If both of these problems are treated, we have found that Prony's method produces excellent estimates from data with high noise levels.

4.4 THE TARGET RECOGNITION PROBLEM

The target recognition problem can be broken into roughly four subproblems:

1. data acquisition,
2. preprocessing of the data,
3. data transformation or characterization, and
4. classification.

We have already discussed the data acquisition problem. The preprocessing problem might consist of choosing which portions of the waveform to use, filtering the waveform, decimating the waveform, or other similar procedures. The preprocessing is intended to condition the data

so that the data transformation procedure can be applied efficiently. The data transformation problem can be handled in different ways. One means of transforming the data may be to apply a resonance extraction procedure to transform the information into a set of poles and residues. After the data are transformed, we must decide to which class the data belongs. In the target recognition problem the classes are defined by the pole sets of known targets. The classification problem then consists of choosing which pole set, if any, in the library of pole sets, best fits the observed radar return.

The data transformation problem need not be treated as a resonance extraction problem. Several other approaches to this problem were studied by Miller [26] and include:

1. Using a linear predictor from a library of linear predictors (each predictor corresponds to a pole set of a known target) to predict the next values of the observed waveform and then using the mean-squared error between the observed waveform and the predicted waveform as a measure of the match [2].

2. Fitting each pole set in the library to the observed waveform by choosing residues to minimize the mean-squared error between the modeled waveform and the observed waveform and then using the mean-squared error as a measure of the match.

3. Correlating the observed waveform with a library of waveforms and using the correlation coefficient as a measure of the match.

At the present time, it appears that fitting each pole set in the library to the observed waveform (the second approach enumerated above) is the most reliable and easily implemented of all. The processing needed for this particular approach is small compared to what might be needed for the application of a resonance extraction procedure. In addition an optimal least-squares fit to the data can always be obtained by using linear, non-iterative methods. An automatic procedure using this approach might consist of the following steps:

1. Data acquisition.
2. Discarding portions of the waveform that do not contain significant amounts of energy.
3. Fitting each pole set in the library to the data.
4. Assigning probability values, P_i, according to how well the i^{th} pole set fits the observed data.

P_i is the probability that the actual target is the i^{th} known target in the library. The rule by which the probabilities are assigned would most likely be developed by a training process where the target identification system observes variations in the measure of fit of each pole set to a known target at various ranges and orientations. The training process could continue even after the system becomes operational by using alternate means of identifying the observed targets. If all the probabilities are sufficiently low then the observed target is declared an "unknown target".

Further study is needed, however, before a firm conclusion can be made about which approach is the best. In particular, attention should be given to which procedures are most efficient with regard to the mode in which they are implemented, e.g., hardware, software, parallel processing, or sequential processing, etc. Such questions will be studied further in Phase II of the current contract.

4.5 THE TARGET INTELLIGENCE PROBLEM

Like the recognition problem, the target intelligence problem can be broken into subproblems:

1. Data acquisition
2. Preprocessing
3. Order selection
4. Construction of resonance estimates
5. Estimating physical features of the target.

But unlike the recognition problem, there is no library of information to work from. The data acquisition prob' has already been discussed. In the following paragraphs we describe the most probable form that an automatic resonance extraction system will take based on our present understanding of the resonance extraction problem.

The ultimate aim of the intelligence problem is to gain some kind of useful information about what the unknown target is. For the purposes of this report, we must assume that this information is going to be derived from the poles. Then the intelligence problem considered in this report requires that poles be extracted from the unknown target's radar return. This could possible be done in real time. However, most likely, a waveform would be stored and processed at a later time. Perhaps the stored waveform would consist of an average of all the returns observed by the target identification system for the unknown target. Regardless of the manner in which the waveform is constructed, some automatic procedure to perform pole extraction is required.

The preprocessing step for a Prony-type resonance extraction procedure might consist of the following:

1. Discard portions of the waveform that do not contain significant amounts of energy.
2. Fourier-transform (FFT) the waveform to determine if the waveform is densely sampled.
3. Perform necessary adjustments to correct the dense sampling problem if it exists.

The order-selection and resonance extimation subproblems can be handled using the EVD of the data matrix, the ML procedure for selecting the order, and Henderson's procedure for eliminating the extraneous resonances. Certain unanswered questions about Henderson's procedure and the ML procedure should be explored before the exact form of these procedures is set.

The fifth subproblem will be studied in Phase II of the present contract. This subproblem consists of establishing a relationship between the pole patterns and the physical characteristics of the target. At the present time very little is known about this subproblem.

4.6 SUMMARY AND CONCLUSIONS

Some of the more significant results of this selection are summarized below:

1. Guidelines have been established for selecting the optimal sampling rate and period of observation for measurement of a transient waveform that is to be used in resonance extraction procedures.

2. Two independent factors effecting the accuracy of estimates of Prony-type procedures have been identified. These factors are "dense sampling" and "bias". The effects of dense sampling can be alleviated by preprocessing techniques or by alternate means of filling the data matrix. The bias problem can be ameliorated by proper construction of the parameter vector (Henderson's procedure).

3. Two distinct problems fall within the target identification definition: target recognition and target intelligence. Each of these problems must be treated in a unique fashion.

4. Tentative forms for procedures and systems to handle both the target recognition problem and the target intelligence problem have been defined. Phase II of the current contract will further refine these tentative plans.

5.0 CONCLUSIONS AND FUTURE EFFORTS

5.1 SUMMARY OF VOLUME I

At the end of each section, except the Introduction, a summary was provided for the results of that section. Below we summarize the conclusions that will serve to determine the course of further efforts.

1. Iterative techniques are not considered for the automatic resonance extraction application because of their inherent problems such as lack of convergence and computational expense.

2. The various noniterative estimation procedures can be interpreted as different methods of combining the eigenvectors of the EVD of the data. There are certain "preferred" methods of combining the eigenvectors of which Henderson's procedure is an example for the case $n > k$. The "preferred" methods are thought to produce unbiased estimates of the coefficients of Prony's difference equation although no proof is available.

3. The ML procedure for selecting the model order is capable of selecting the proper order without any knowledge of the noise level whatsoever by simply observing the eigenvalues of the data.

4. Guidelines have been established for selecting the optimal sampling rate and period of observation for measurement of a transient waveform that is to be used in resonance extraction procedures.

5. Methods for automatically sensing the problem of dense sampling and for allaying its effects have been proposed.

6. Tentative forms for procedures and systems to handle both the target recognition problem and the target intelligence problem have been defined.

5.2 CURRENT STATUS OF AND UNANSWERED QUESTIONS ABOUT RESONANCE EXTRACTION METHODS

At the present time it is possible to construct automated, efficient procedures to:

1. Estimate the number of resonances present in the data, and
2. Construct estimates of the "true" resonances that in some sense fit the original data,

with no knowledge of the noise given whatsoever. These procedures should be quite tolerant to noise. The procedures are based on the eigenvalue analysis of the data.

The maximum likelihood procedure for order selection, described in Section 3, will perform the first step. Henderson's procedure, also described in Section 3, will perform the second step. Although these procedures perform adequately at this time, there are some unanswered questions, that if addressed, could create new ways to further improve performance.

One unanswered question relates to the statistics of eigenvalues for alternate forms of the data matrix that might be used to allay the effects of dense sampling. If the eigenvalue statistics do change significantly, then another model for the statistics must be developed for the alternate forms of the data matrix and used in the ML procedure for order selection. Another question relates to the proof that EVD allows unbiased estimation of the coefficients of Prony's difference equation. Such a proof is needed to place the use of EVD techniques on firm ground.

Other unanswered questions about Henderson's procedure which were posed in Section 3 are:

1. How should the weakest eigenvectors be weighted prior to the use of this procedure?
2. What specific procedures can be applied in place of Gaussian elimination in this procedure?

Further study along the lines of these questions will almost certainly yield further improvements in the procedures. Perhaps the most important unanswered question is how well the methods for preprocessing, order selection, and resonance estimation will perform with real measured data.

5.3 FUTURE EFFORTS

Under Phase II of the current contract the concepts and procedures developed in Phase I will be further developed and plans for specific non-cooperative target recognition (NCTR) systems will be further refined.

To accomplish the above the following questions must be addressed:

1. What is the relationship between the natural resonances and the physical shape and dimensions of a target?
2. What forms can a NCTR system assume? Which form is the best under what conditions?
3. What modifications are required if the procedures are to be implemented in hardware?

Answers to these questions will be pursued by simulating possible NCTR systems with measured data in the second phase of this contract.

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7.0 LIST OF SYMBOLS

Roman Symbols:

A_j	-	true residues
$\hat{A}_j$	-	estimated residues
B	-	a constant that determines the value of y that maximizes f_1
C_i	-	coefficients of a linear combination of the eigenvectors
D	-	integer that specifies how many times the sampling interval can be increased without aliasing any of the frequencies of interest
d	-	M-dimensional equation error vector
d_i	-	i^{th} element of the equation error vector
d_{NQE}	-	particular equation error vector that satisfies $Qx_{NQE} = d_{NQE}$
$\bar{E}$	-	error matrix defined as $\bar{Q}-\bar{W} = \bar{E}$
e_i	-	measurement error of the i^{th} sample
F_j	-	transfer function of the j^{th} filter in the generalized model
f	-	frequency function for y
f_1	-	approximation to the frequency function f
G	-	$(\ell \times N)$-dimensional matrix whose rows span the ℓ^{th} autoregression nullspace.
g	-	equation error vector for the generalized model

g_i	-	i^{th} element of the equation error vector for the generalized model
H	-	$(\ell'' \times k' + 1)$-dimensional matrix used in Henderson's procedure.
H^m	-	estimate of the transfer function of the single-input, single-output system which models the scatterer
h	-	any N-dimensional vector in the row space of G_ℓ
h_j	-	j^{th} element of h
I	-	an identity matrix of appropriate dimension
i	-	general index
j	-	general index
K	-	constant that determines the amplitude of f_1
k	-	true number of poles
k'	-	estimated number of poles
k'	-	estimate of the true order k
L_c	-	cut-off normalized mean log-likelihood used in the maximum likelihood procedure for selecting order
L_ℓ	-	normalized mean log-likelihood function
L_o	-	expected normalized mean log-likelihood for pure noise
ℓ	-	index associated with the lower eigenvalues
M	-	number of rows of Q, M = m + 1
m	-	equals M-1 (defined for convenience)
N	-	number of columns of Q, N = n + 1

N_M	-	number of samples in the measured waveform
N_s	-	number of samples in the analyzed waveform
n	-	number of poles modeled
P	-	transpose product of Q or Q^TQ
P_i	-	the probability that the actual radar target is the i^{th} known target in the target library
P_s	-	Any (N x N)-dimensional singular matrix
$\hat{P}_s$	-	(N x N)-dimensional singular approximant to P
p_{ji}	-	the i^{th} sample of the output of the j^{th} filter on the excitation in the generalized model
Q	-	(M x N)-dimensional measured data matrix
$\bar{Q}$	-	(M x n)-dimensional version of Q or Q with the N^{th} column removed.
Q_A	-	an alternate form of Q that is useful in treating the dense-sampling problem
Q_s	-	the n-rank approximant to Q
$\bar{q}$	-	the N^{th} column of Q
q_i	-	the i^{th} sample of measured response of the linear system
q_{ji}	-	the i^{th} sample of the output of the j^{th} filter on the response in the generalized model
r	-	the (M + N-1)-dimensional residual vector
r_i	-	the i^{th} element of r
S	-	(M x N)-dimensional diagonal matrix whose diagonal elements are non-negative and are called the singular values of Q
s_j	-	true s-plane poles
s'_j	-	estimated s-plane poles

S_s - the (M x N)-dimensional matrix constructed from S by forcing the smallest singular value to zero.

s - Laplace - transform variable

T - period of observation (length of waveform in time)

T_{max} - expected period of time the measured transient waveforms are above the noise

T_{min} - one-half cycle of the lowest frequency of of interest, equals π/ω_{Low}

t - time variable

Δt - time step

U - (M x M)-dimensional orthogonal matrix whose columns consist of the eigenvectors of QQ^T

V - (N x N)-dimensional orthogonal matrix whose columns are the eigenvectors of Q^TQ

V^W - the (N x N)-dimensional orthogonal matrix whose columns consist of the eigenvectors of W^TW

v_i - the i^{th} eigenvector of Q^TQ corresponding to λ_i or the i^{th} column of V.

v_{ij} - the element of V belonging to the i^{th} row and j^{th} column

W - (M x N)-dimensional true data matrix

$\bar{w}$ - uncorrupted version of $\bar{q}$

w_i - uncorrupted value of q_i

X - matrix constructed from the coefficients of Prony's difference equation

x - any N-dimensional parameter vector or vector of coefficients at Prony's difference equation

$\bar{x}$ - any n-dimensional version of x or x with the N^{th} element removed

$\bar{x}_I$ - solution of the inhomogeneous equation, $\bar{\bar{Q}}^T\bar{\bar{Q}}\bar{x}_I = \bar{\bar{Q}}^T\bar{q}$

x_I	-	scaled parameter vector constructed from $\bar{x}_I$ which yields identical pole estimates
$\bar{x}_{IE}$	-	expected inhomogeneous solution
x_{NQE}	-	weakest eigenvector of Q^TQ
$\bar{x}_{NQE}$	-	n-dimensional vector constructed from x_{NQE} by scaling x_{NQE} so that its N^{th} element is one and then eliminating the N^{th} element.
x_J	-	parameter vector constructed with Jain's method
x^{ℓ}	-	any vector in the space spanned by the ℓ weakest eigenvectors of Q^TQ
Y	-	(N x N)-dimensional matrix whose diagonal elements consist of samples of the random variable y
y	-	a random variable used to model the random component in the eigenvalues due to noise in the data
z	-	z-transform variable
z_j	-	true z-plane poles
$\hat{z}_j$	-	estimated z-plane poles

Greek Symbols:

α_j	-	coefficients of Prony's difference equation
β_j	-	scaling factor for the output of the j^{th} filter on the excitation in the generalized model
Δ_{ij}	-	element of the i^{th} row and j^{th} column of adj Q^TQ
δ	-	cut-off parameter used in the likelihood ratio criterion for selecting the order
θ	-	2N-dimensional parameter vector for the generalized model
Λ_N	-	(N x N)-dimensional diagonal matrix whose diagonal elements are the eigenvalues of Q^TQ
Λ_M	-	(M x M)-dimensional diagonal matrix whose diagonal elements are the eigenvalues of QQ^T
Λ_{NS}	-	diagonal (N x N)-dimensional matrix constructed from Λ_N by forcing the smallest diagonal element to zero
Λ_N^W	-	(N x N)-dimensional diagonal matrix whose diagonal matrix whose diagonal elements are the eigenvalues of W^TW
λ_i	-	i^{th} eigenvalue of Q^TQ the eigenvalues are ordered $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$
λ_i^W	-	i^{th} eigenvalue of W^TW, the eigenvalues are ordered $\lambda_1^W \leq \lambda_2^W \leq \ldots \leq \lambda_N^W$
ξ	-	expectation operator
σ_{est}	-	adjustable parameter used to form the maximum likelihood estimate of σ from the lower eigenvalues of Q^TQ
σ	-	the standard deviation of the assumed Gaussian-distributed, zero-mean, and uncorrelated noise corrupting the samples of the waveform

Ω	-	the (M x 2N)-dimensional data matrix for the generalized model
ω	-	sampling rate (angular frequency)
ω_{HIGH}	-	highest frequency of interest
ω_{LOW}	-	lowest frequency of interest
ω_{max}	-	Maximum feasible sampling rate
ω_{min}	-	Nyquist frequency, equals $2\omega_{HIGH}$

Miscellaneous symbols:

$\|\|A\|\|$	-	denotes the euclidean norm of vector A or matrix A
∇_x	-	gradient operator with respect to the vector x
adjA	-	denotes the adjoint matrix of matrix A or the transpose of the matrix of cofactors of A
detA	-	denotes the determinant of matrix A
ℓnA	-	denotes the natural logarithm of some real or complex number A
exp (A)	-	denotes e, the base of natural logarithms, raised to the power A
A^{-1}	-	denotes the inverse of square matrix A
A*	-	denotes the transpose conjugate of complex matrix A